ETBS204-MODULE – IV

ANALYTIC FUNCTIONS

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4 Analytic Functions

4.1 Introduction

You must have studied earlier, that the quadratic $ax^2 + bx + c = 0$ has its roots real, equal or imaginary (complex) according as its discriminant $b^2 - 4ac$ is greater equal or less than zero. We also know that square of any real number is never negative and as such our real number system fails to give the solution of the equations of the type $x^2 + 1 = 0$ or $x^2 - 4x + 7 = 0$. We take $i = \sqrt{-1}$ or $i^2 = -1$ and then introduce $z = a + ib$, where a and b are real numbers and $a + ib$ is called a *complex number*; then a is called the *real part* of z and b the *imaginary part* of z which are denoted by $a = Re(z), b = Im(z)$.

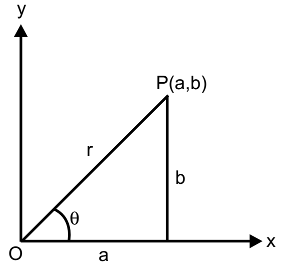
Two complex numbers are said to be equal i.e., $x + iy = a + ib$. if and only if $x = a, y = b$.

If we change the sign of the imaginary part of complex number z, then we obtain another complex number which is called *conjugate complex* of the given complex number and is denoted by $\overline{z}$. Thus if $z = a + ib$ then $\overline{z} = a - ib$.

Clearly $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2$ which is purely real also $\dfrac{z + \overline{z}}{2} = a =$ real part of $z = Re(z)$; $\dfrac{z - \overline{z}}{2i} = b =$ imaginary part of $z = Im(z)$.

Modulus and Amplitude: Polar form of Complex Number

Let $P(a, b)$ be a point in the complex place corresponding to complex number $z = a + ib$, so that $a = r\cos\theta$ and $b = r\sin\theta$.



Squaring and adding and also dividing, we get, $r^2 = a^2 + b^2 \quad r = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$ or $\theta = \tan^{-1}\dfrac{b}{a}$

The quantity r is called *modulus* or absolute value of complex number z and is denoted by $|z|$ and the quantity θ is called its *amplitude*.

$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called polar form of complex number.

The XOY - plane, in which the points represent the complex number, is called the complex plane or *Argand plane* or *Argand diagram*.

Fundamental operations of algebra

Addition

$$\begin{aligned} z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2) \\ &= (a_1 + a_2) + i(b_1 + b_2) \end{aligned}$$

Subtraction

$$\begin{aligned} z_1 - z_2 &= (a_1 + ib_1) - (a_2 + ib_2) \\ &= (a_1 - a_2) + i(b_1 - b_2) \end{aligned}$$

Multiplication

$$\begin{aligned} z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= a_1a_2 + i(a_1b_2 + a_2b_1) + i^2 b_1b_2 \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \end{aligned}$$

Division

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + ib_1}{a_2 + ib_2} \\ &= \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2} \\ &= \left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{b_1a_2 - b_2a_1}{a_2^2 + b_2^2}\right) \end{aligned}$$

The Complex Variable

The quantity $z = x + iy$ where x and y are two independent real variables is called a *complex variable*. The Argand plane in which the variable z is represented by the points is called z - plane. The point that represents the complex variable z is point z.

Neighbourhood of a Point

Let z_0 be a point in the Argand diagram. Then the neighbourhood of this point z_0 is defined as the set of all those points z such that $|z - z_0| < \in$, where $\in$ is an arbitrary small positive number. This $\in$ is called the radius of the neighbourhood of z_0.

Deleted Neighbourhood of z_0

If from the neighbourhood of a point z_0 defined by $|z - z_0| < \in$, we exclude the point z_0, then such a neighbourhood is called the *deleted neightbourhood* of the point z_0 i.e.

$$0 < |z - z_0| < \in$$

Neighbourhood of the Point at Infinitely

The set of all points z such that $|z| > K$ where K is any positive real number is called a *neighbourhood* of the point at infinity.

Function of a Complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in a certain portion of the complex plane also called as the domain R of the complex plane there corresponds one or more values of w then w is said to be function of z and is written as $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u(x, y)$ and $v(x, y)$ are real functions of the real variables x and y.

Single Valued and Many Valued Function

If for every point z in a region R of the z - plane there corresponds a unique value for w, then w is called a *single valued function* of z.

If more than one value of w corresponds to a point z in a region R of the z - Plane, then w is said to be a *many (multiple) valued function* of z in that region.

For example

$$w = f(z) = z \text{ is a single valued function of } z,$$

$$\text{but} \quad w = f(z) = \sqrt{z} \text{ is a many valued function of } z$$

Example 4.1

Express the following function of z in the form $u + iv$.

(i) $w = \cos z$ (ii) $w = \tan z$ (iii) $w = \sec z$

(iv) $w = e^z$ (v) $w = \log z$ (vi) $w = \dfrac{1}{z}$

Solution:

(i)
$$\begin{aligned} w &= \cos z \\ &= \cos(x + iy) \\ &= \cos x \cosh y - i \sin x \sinh y \end{aligned}$$

$[\because \cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos iy = \cos hy$

$\sin iy = i \sin hy]$

$$\therefore \quad u = \cos x \cosh y \text{ and } v = -\sin x \sinh y$$

$$\therefore \quad w = f(z) = u(x, y) + iv(x, y)$$

(ii)
$$\begin{aligned} w = \tan z &= \tan(x + iy) \\ &= \frac{\sin(x + iy)}{\cos(x + iy)} \\ &= \frac{2\sin(x + iy)\cos(x - iy)}{2\cos(x + iy)\cos(x - iy)} \end{aligned}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)} \qquad \because \begin{pmatrix} \sin iy = i \sinh y \\ \cos iy = \cosh y \end{pmatrix}$$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \left(\frac{\sinh 2y}{\cos 2x + \cosh 2y} \right)$$

$$\therefore \qquad u = \frac{\sin 2x}{\cos 2x + \cosh 2y} \text{ and } v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

(iii) $w = \sec z = \sec(x + iy)$

$$= \frac{1}{\cos(x+iy)}$$

$$= \frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2[\cos x \cos iy + \sin x \sin y]}{\cos(2x) + \cos(2iy)}$$

$$= \frac{2\cos x \cos y + i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$$\therefore \quad u = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y} \text{ and } v = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

(iv) $w = e^z$

$$= e^{(x+iy)}$$

$$= e^x \cdot e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x \cos y + i e^x \sin y$$

$$\therefore \quad u = e^x \cos y \text{ and } v = e^x \sin y$$

(v) $w = \log z = \log re^{i\theta} = \log r + i\theta$

$$= \log(x^2 + y^2)^{1/2} + i \tan^{-1} \frac{y}{x}$$

$$= \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1} \frac{y}{x}$$

$$\therefore \qquad u = \frac{1}{2}\log(x^2 + y^2) \text{ and } v = \tan^{-1}\left(\frac{y}{x}\right)$$

(vi) $w = \frac{1}{z}$

$$= \frac{1}{x + iy}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$
$$= \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$
$$\therefore \quad u = \frac{x}{x^2+y^2} \text{ and } v = -\frac{y}{x^2+y^2}$$

Limit of a Function of a Complex Variable

Let $w = f(z)$ be any single valued function defined in the deleted neighbourhood of $z = a$. We say that $f(z)$ tends to limit ℓ as z tends to a along any path in a defined region, if to each positive arbitrary number $\in$, how ever small there corresponds a positive number depending upon $\in$ such that $|f(z) - \ell| < \in$ for all points of the region for which $0 < |z-a| < \delta$.

Symbolically, we write

$$\underset{z\to a}{\text{Lt}}\ f(z) = \ell$$

Continuity

A function $w = f(z)$ of a complex variable z defined for a certain region D is said to be continuous at the point $z = a$ of D, if given a positive number $\in$, we can find a number $\delta(\delta > 0)$ depending on $\in$ such that $|f(z) - f(a)| < \in$ for all points z of D satisfying the condition $0 < |z-a| < \delta$.

i.e. $f(z)$ will be continuous at $z = a$, if

$$\underset{z\to a}{\text{Lt}}\ f(z) = f(a)$$

Also if $f(z) = u(x,y) + iv(x,y)$ then $f(z)$is continuous if and only if $u(x,y)$ and $v(x,y)$ are separately continuous functions of x and y.

Example 4.2

Show that the function $f(z) = \dfrac{\bar{z}}{z}$ does not have a limit as $z \to 0$.

Solution:

$$f(z) = \frac{\bar{z}}{z} = \frac{(x-iy)}{(x+iy)}$$

Suppose $z \to 0$ along the path $y = mx$

$$\text{Along this path } f(z) = \frac{x-imx}{x+imx}$$
$$= \frac{1-im}{1+im} \qquad \text{as } x \neq 0$$

Hence if $z \to 0$ along the path $y = mx, f(z)$ tends to $\dfrac{1-im}{1+im}$ which is different for different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Example 4.3

State the basic difference between the limit of a function of a real variable and that of a complex variable. **(AU 2012)**

Solution:

In real variable $x \to x_0$ means x approaches x_0 along x axis (or) a line parallel to $x-$axis.

In complex variable $z \to z_0$ means z approaches z_0 along any path joining z and z_0 in $z-$plane.

Example 4.4

If $f(z) = \dfrac{x^2+3y^2}{(x+y)^2}$, then show that $\lim\limits_{z \to z_0} f(z)$ does not exist.

Solution:

If $\lim\limits_{z \to z_0} f(z)$ exists, then the value of $f(z)$ is sufficiently close to a unique value when z is sufficiently close to 0.

Let $z = x + iy = a + ima$, where a is a very small value close to zero. As $z \to 0 \;\; a \to 0$ in the limit.

$$\therefore \quad x = a \;\text{ and }\; y = ma$$

$$f(z) = \frac{x^2+3y^2}{(x+y)^2} = \frac{a^2+3m^2a^2}{(a+ma)^2} = \frac{1+3m^2}{1+m^2}$$

Now taking the limits as $z \to 0$, we have

$$\lim_{z \to 0} f(z) = \lim_{a \to 0} \frac{1+3m^2}{(1+m)^2} = \frac{1+3m^2}{(1+m)^2}$$

This value depends on m and so it is not a fixed or unique value for all z sufficiently close to 0. Hence $\lim_{z \to 0} f(z)$ does not exist.

Example 4.5

Show that $f(x,y) = \dfrac{2xy}{x^2+y^2}$ is discontinuous at (0,0) given that $f(0) = 0$.

Solution:

Here $f(0)$ is defined. We have to show that $\lim_{z\to 0} f(z)$ does not exist. Consider the curve $y = mx$ along which x and y vary and come close to the origin.

Choose $x = a$, where a is a very small and close to 0.

$$\therefore \quad \text{as} \quad z \to 0, a \to 0 \quad \text{in the limit.}$$

$$\therefore \quad f(z) = \frac{2xy}{x^2 + y^2} = \frac{2a(am)}{(a)^2 + (am)^2} = \frac{2a^2m}{a^2 + a^2m^2} = \frac{2m}{1 + m^2}$$

Taking limit as $z \to 0$, we have

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{2m}{1 + m^2} = \frac{2m}{1 + m^2}$$

This value depends on m and so it is not a fixed value for all z sufficient close to 0.

Therefore $\lim\limits_{z\to 0}$ does not exist and hence $f(z)$ is not continuous at $z = 0$.

Example 4.6

If $f(z) = \dfrac{x^3y(y - ix)}{x^6 + y^2}, z \neq 0, f(0) = 0$, prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Solution:

Now $y - ix = -i^2y - ix = -i(x + iy) = -iz$

$$\therefore \qquad f(z) = -\frac{x^3yiz}{x^6 + y^2}, \quad f(0) = 0$$

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} -\frac{ix^3y}{x^6 + y^2}$$

now if $z \to 0$ along any radius vector say $y = mx$ then

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{x\to 0} \frac{-x^4mi}{x^6 + m^2x^2} = \lim_{x\to 0} \frac{-imx^2}{x^4 + m^2} = 0$$

Now let as suppose that $z \to 0$ along the curve $y = x^3$ then

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{x\to 0} \frac{-x^6i}{x^6 + x^6} = \frac{-i}{2} \neq 0$$

4.2 Derivative of a Complex Function

A function $f(z)$ is said to be differentiable at z_0 if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists. This limit is called the *derivative* of $f(z)$ at z_0 and it is denoted as $f'(z)$.

On putting $z - z_0 = \Delta z$, we have,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f'(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{when limit exists}$$

The function $f(z)$ is said to be differentiable at z if limit exists.

Analytic Function

The function $f(z)$ is said to be *analytic* at a point z_0 if $f(z)$ is differentiable at z_0 and at every point in some neighborhood of z_0.

A function $f(z)$ is said to be analytic in a region R of the z-plane if it is analytic at every point of R.

The terms *regular and holomorphic* are also sometimes used as synonyms for analytic.

■ **Note:** A point, at which a function $f(z)$ is not analytic is called a singular point or singularity of $f(z)$.

4.3 Cauchy - Riemann Equations

Theorem: Necessary conditions for $f(z)$ to be analytic

The necessary condition of $w = f(z) = u + iv$ to be analytic at any point z of its region R is that the four partial derivatives u_x, u_y, v_x and v_y should exist and satisfy the equations $u_x = v_y$ and $u_y = -v_x$

$$\text{i.e.} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof: Let $f(z) = u(x, y) + iv(x, y)$ be analytic at any point z of its region

$$\therefore\ f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{1}$$

exists and is unique ie it is independent of the path along which $\Delta z \to 0$.

$$\begin{aligned} \text{Now} \quad z &= x + iy \\ \therefore \quad \Delta z &= \Delta x + i\Delta y \\ f(z) &= u(x, y) + iv(x, y) \\ f(z + \Delta z) &= u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \end{aligned}$$

Thus (1)

$$f'(z) = \lim_{\substack{\Delta z \to 0 \\ \Delta y \to 0}} \frac{\begin{array}{c}[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] \\ -[u(x,y)+iv(x,y)]\end{array}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right] \tag{2}$$

must exist uniquely. Now we evaluate the limit in two different ways.

□ **Case: 1** Take Δz to be purely real i.e. $\Delta z = \Delta x, \Delta y = 0$ and $\Delta x \to 0$

Hence from (1) we get

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{[u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{[u(x+\Delta x, y) - u(x,y)]}{\Delta x}\right] + i \lim_{\Delta x \to 0} \left[\frac{[v(x+\Delta x, y) - v(x,y)]}{\Delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \tag{3}$$

Since $f'(z)$ exists therefore the above limit exists i.e. u_x and v_x exists.

□ **Case: 2** Taking Δz to be purely imaginary and hence $\Delta z = i\Delta y$, $\Delta x = 0$

Hence from (2), we get

$$f'(z) = \lim_{\Delta y \to 0} \left[\frac{[u(x, y+\Delta y) - u(x,y)]}{i\Delta y}\right] + i \lim_{\Delta y \to 0} \left[\frac{[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}\right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -iu_y + v_y \tag{4}$$

Since $f'(z)$ exists there fore the above limit exists i.e. u_y and v_y exist.

Also by definition we know that the limit should be unique and hence the two limits obtained in (3) and (4) should be identical

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = -\frac{i\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The above equations are known as *Cauchy - Riemann equations* and they may be written as

$$u_x = v_y \quad \text{and} \quad u_y \;\; - v_x$$

■ **Note:**

1. The Cauchy -Riemann equations are referred as C-R equations
2. The function $f'(z)$ is given by any one of the following

$$f'(z) = u_x + iv_x; f'(z) = u_x - iu_y$$
$$f'(z) = v_y - iu_y; f'(z) = v_y + iv_x$$

3. If $w = f(z)$ then $f'(z)$ is also denoted by $\dfrac{dw}{dz}$

$$\text{Thus} \quad \frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

4. The $C-R$ equations are only the necessary conditions for a function $f(z)$ to be analytic i.e. If the given function $f(z)$ is analytic it will satisfy the $C-R$ equations. Conversely if a function $f(z)$ satisfies the $C-R$ equations the function need not be analytic i.e. $C-R$ equations are not sufficient for a function to be analytic.

Theorem : Sufficient conditions for f(z) to be analytic

The function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a region R if the four partial derivatives u_x, u_y, v_x and v_y

1. exist
2. they are continuous
3. they satisfy the $C-R$ equations namely $u_x = v_y$ and $v_y = -v_x$ at every point of R.

Proof: Now $w = u + iv \qquad \therefore \;\; \Delta w = \Delta u + i\Delta v$

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$$
$$= \left.\begin{array}{c} [u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)] \\ +[u(x, y + \Delta y) - u(x, y)] \end{array}\right\} \qquad (1)$$

By mean value theorem we know that if $f(x)$ is continuous in $a \leq x \leq b$ and differentiable in $a < x < b$ then $f(a + h) - f(a) = hf'(a + \theta h)$.

Where $0 < \theta < 1$

Applying the result in (1) we get

$$\Delta u = \Delta x \cdot u_x(x + \theta\Delta x, y + \Delta y) + \Delta y \cdot u_y(x, y + \theta^2\Delta y) \qquad (2)$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$

Again u_x and u_y are given to be continuous

$$\therefore \quad |u_x(x + \theta_1\Delta x, y + \Delta y) - u_x(x, y)| < \in$$
$$|u_y(x, y + \theta_2\Delta y) - u_y(x, y)| < \eta$$

now choosing $\in_1 < \in$ and $\eta_1 < \eta$ we have from above.

$$u_x(x + \theta_1\Delta x, y + \Delta y) - u_x(x, y) = \in_1$$
$$u_y(x, y + \theta_2\Delta y) - u_y(x, y) = \eta_1$$

Hence from (2) by the help of above relation, we get

$$\Delta u = (u_x(x, y) + \in_1)\Delta x + (u_y(x, y) + \eta_1)\Delta y \qquad (3)$$

Similarly, we get

$$\Delta y = (v_x(x, y) + \in_2)\Delta x + (v_y(x, y) + \eta_2)\Delta y \qquad (4)$$

Now

$$\Delta w = \Delta u + i\Delta v$$
$$= [(u_x + \in_1)\Delta x + (u_y + \eta_1)\Delta y)] + i[(v_x + \in_2)\Delta x + (v_y + \eta_2)\Delta y]$$

$$\text{or} \quad \Delta w = \left.\begin{array}{c}(u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\in_1 + i \in_2) \\ \Delta x + (\eta_1 + i\eta_2)\Delta y\end{array}\right\} \qquad (5)$$

now by C-R equations $u_x = v_y$ and $u_y = -v_x = i^2 v_x$ and choosing $\in_3 = \in_1 + \in_2$ and $\eta_3 = \eta_1 + \eta_2$

Hence (5) can be written as

$$\Delta w = (u_x + iv_x)\Delta x + i(iv_x + u_x)\Delta y + \in_3 \Delta x + \eta_3\Delta y$$
$$= (u_x + iv_x)(\Delta x + i\Delta y) + \in_3 \Delta x + \eta_3\Delta y$$

dividing throughout by $\Delta z = \Delta x + i\Delta y$, we get

$$\frac{\Delta w}{\Delta z} = u_x + iv_x + \frac{\in_3 \Delta x + \eta_3\Delta y}{\Delta x + i\Delta y}$$

Taking limit when $\Delta z \to 0$ so that $\Delta x \to 0$ $\Delta y \to 0$ $\in_3 \to 0$ $\eta_3 \to 0$, we get

$$f'(z) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + iv_x$$

since u_x, v_x exist and are unique.

$\therefore$ $f'(z)$ exist i.e. the derivative of $w = f(z)$ exists at every point in R. Hence $w = f(z)$ is analytic in the region R.

Polar Form of Cauchy-Riemann Equations

Theorem: If the function $w = f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in a region R then the partial derivation $\dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}, \dfrac{\partial v}{\partial r}$ and $\dfrac{\partial v}{\partial \theta}$ must exist and satisfy the $C - R$ equations $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{r\theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ at every point in that region.

Proof: Given $f(z) = u(r, \theta) + iv(r, \theta)$ where $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Then $\Delta z = \Delta(re^{i\theta})$

$$\therefore \quad \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\begin{array}{c} u(r+\Delta r, \theta+\Delta\theta) + iv(r+\Delta r, \theta+\Delta\theta) \\ -u(r,\theta) - iv(r,\theta) \end{array}}{\Delta(re^{i\theta})}$$

$$\text{or} f'(z) = \lim_{\Delta z \to 0} \left[\begin{array}{c} \dfrac{u(r+\Delta r, \theta+\Delta\theta) - u(r,\theta)}{\Delta re^{i\theta}} \\ +i\dfrac{v(r+\Delta r, \theta+\Delta\theta) - v(r,\theta)}{\Delta re^{i\theta}} \end{array} \right]$$

Given that $f(z)$ is analytic, $f'(z)$ is unique in whatever manner $\Delta z \to 0$.

□ **Case: 1** When $\Delta z \to 0, \Delta r \to 0$ if θ is kept fixed

Then $\Delta z = \Delta(re^{i\theta}) = e^{i\theta}\Delta r$

$$\therefore \; f'(z) = \lim_{\Delta z \to 0} \frac{1}{e^{i\theta}} \left[\frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r} + i\frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r} \right]$$

$$= e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \tag{1}$$

□ **Case: 2** When $\Delta z \to 0, \Delta\theta \to 0$ if r is kept fixed

Then $\Delta z = \Delta(re^{i\theta}) = re^{i\theta} i\Delta\theta$

$$\therefore \; f'(z) = \lim_{\Delta\theta \to 0} \left[\frac{u(r, \theta+\Delta\theta) - u(r,\theta) + i(v, \theta+\Delta\theta) - v(r,\theta)}{re^{i\theta} i\Delta\theta} \right]$$

$$= \frac{1}{r}e^{-i\theta} \lim_{\Delta\theta \to 0} \left[\frac{1}{i}\frac{u(r, \theta+\Delta\theta) - u(r,\theta)}{\Delta\theta} + \frac{v(r, \theta+\Delta\theta) - v(r,\theta)}{\Delta\theta} \right]$$

$$= \frac{1}{r}e^{-i\theta}\left[\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{r}e^{-i\theta}\left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right] \tag{2}$$

since $f'(z)$ exists i.e. $\dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}, \dfrac{\partial v}{\partial r}, \dfrac{\partial v}{\partial \theta}$ exists at every point in R. Since $f'(z)$ exists uniquely. From (1) and (2)

$$e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right)$$

Equating the real and imaginary parts, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

These are the Cauchy - Riemann equations in polar coordinates.

4.4 Properties of Analytic Functions

I. Harmonic property

The real and imaginary parts of an analytic function $w = u+iv$ satisfy the Laplace equations in two dimensions i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Proof: Since $w = u + iv$ is analytic in a region R of the z-plane, u and v satisfy the C-R equations namely

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$\text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

If u and v are assumed to have continuous second order partial derivations in the region R, then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ *and* $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$.

Differentiating both sides of (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

differentiating both sides of (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \tag{4}$$

adding (3) and (4) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\text{i.e.,} \qquad u_{xx} + u_{yy} = 0$$

i.e. the real part u satisfies Laplace equation. Similarly, differentiating (1) partially with respect to y and (2) partially with respect to x and adding, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{i.e.} \quad v_{xx} + v_{yy} = 0$$

i.e. the imaginary part v satisfies Laplace equation.

■ **Note:**

1. Any function of x and y having continuous partial derivatives of first and second order and also satisfying Laplace's equation is called a *Harmonic function*. Hence u and v are *Harmonic function* if $f(z) = u + iv$ is analytic.
2. If u and v are harmonic functions such that $u + iv$ is analytic then each is called the *conjugate harmonic function* of the order in the region R.
3. If $f(z) = u + iv$ is an analytic function then u and v are harmonic functions however if $u \& v$ are any two harmonic function then $f(z) = u + iv$ need not be an analytic function.

II. The real and imaginary parts of an analytic function $w = u(r, \theta) + iv(r, \theta)$ satisfy the Lapalce equation in polar coordinates.

Proof: Since $w = u(r, \theta) + iv(r, \theta)$ is analytic in a region R of the z-plane u and v satisfy $C - R$ equation in polar coordinates.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{r\theta} \qquad (1)$$

$$\text{and} \quad \frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{r\theta} \qquad (2)$$

Differentiating (1) partially with respect to r, we get

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial u}{\partial \theta} \qquad (3)$$

Differentiating (2) partially w.r.t θ, we get

$$\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial^2 v}{\partial \theta \partial r} \qquad (4)$$

Assuming the continuity of the mixed derivatives we get.

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{1}{r}\left(\frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial^2 u}{\partial \theta \partial r}\right) = 0$$

Hence v satisfies Laplace equation in polar coordinates.

III. (Orthogonal property)

If $w = u + iv$ is analytic function the curves of the family $u = c_1$ and the curves of the family $v = c_2$ cut orthogonally, where c_1 and c_2 are constants.

Proof: $u(x, y) = c_1$ represents a family of curves. Consider a representative member of the family $u(x, y) = c_1$, corresponding to $c_1 = c_1'$

$$\text{i.e} \quad u(x, y) = c_1'$$

Taking differentials on both sides, we get

$$du = 0$$

$$\text{i.e} \quad \frac{\partial u}{\partial x}dx + \frac{\partial y}{\partial y}dy = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = m_1, \text{ say,}$$

m_1 is the slope of the curve $u = c_1'$ at (x, y)

Similarly, considering a member of the second family whose equation is $v(x, y) = c_2'$, we get

$$\frac{dy}{dx} = -\frac{(\partial v/\partial x)}{(\partial v/\partial y)} = m_2,$$

where m_2 is the slope of $v(x, y) = c_2'$

$$\text{Now} \quad m_1 m_2 = \frac{(\partial u/\partial x)}{(\partial u/\partial y)} \cdot \frac{(\partial v/\partial x)}{(\partial v/\partial y)}$$

$$= \frac{(\partial u/\partial x)}{(\partial u/\partial y)} \cdot \frac{(-\partial u/\partial y)}{(\partial u/\partial x)} \quad \text{(by C-R equations)}$$

$$= -1.$$

The product of the slopes is equal to -1. This is true at the point of intersection of the two curves $u(x, y) = c_1'$ and $v(x, y) = c_2'$.

Therefore every member of the family $u = c_1$ cuts orthogonally every member of the family $v = c_2$.

■ **Note :**

1. If $f(z) = u(r, \theta) + iv(r, \theta)$ is an analytic function, the curves of the family $u(r, \theta) = c_1$ cut orthogonally the curves of the family $v(r, \theta) = c_2$ where c_1 and c_2 are arbitrary constants.

2. The two families are said to be the *orthogonal trajectories* of each other.

Construction of an Analytic Function Whose Real or Imaginary Part is Known

Method 1

□ **Case 1:** Let $u(x, y)$ the real part of the analytic function $f(z) = u + iv$ be known.

We first find $v(x, y)$. Since $u(x, y)$ is given $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be found out.

$$\begin{aligned} \therefore \quad dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \\ &= \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad (1) \end{aligned}$$

$u(x, y)$ being a harmonic function satisfies

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \therefore \quad \frac{\partial}{\partial x}(u_x) &= \frac{\partial}{\partial y}(-u_y) \end{aligned}$$

$\therefore$ the R.H.S. of (1) is a exact differential integrating both sides of (1), we get

$$v = \int \left[\left(\frac{-\partial u}{\partial y} \right) dx + \left(\frac{\partial u}{\partial x} \right) dy \right] + c$$

where c is an arbitrary constant of integration.

□ **Case 2:** Let $v(x, y)$, the imaginary part of the analytic function $f(z)$ is known. If $u(x, y)$ be the real part , then by $C - R$ equations.

$$\begin{aligned} u_x = v_y \ \ &\text{and} \ \ u_y = -v_x \\ \therefore \quad du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \qquad (2) \end{aligned}$$

$v(x, y)$being harmonic function$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

We get

$$\frac{\partial}{\partial y}(v_y) = \frac{\partial}{\partial x}(-v_x)$$

$\therefore$ R.H.S. of (2) is an exact differential.

$$\therefore \quad u = \int \left[\left(\frac{\partial v}{\partial y} \right) dx + \left(\frac{-\partial v}{\partial x} \right) dy \right] + c$$

where c is an arbitrary constant of integration.

Method II (Milne - Thomson method)

Let $u(x, y)$ be the real part of the analytic function $f(z) = u + iv$ is given

We first find $f'(z)$ as a function of z,

$$
\begin{aligned}
\therefore \quad f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{(since } u \text{ is given} \frac{\partial u}{\partial x} \text{and } \frac{\partial u}{\partial y} \text{can be found out)} \\
&= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad (\because \text{ by C-R equations}) \\
&= \frac{\partial u}{\partial x}(x, y) - i\frac{\partial u}{\partial y}(x, y) \\
&= \frac{\partial u}{\partial x}(z, 0) - i\frac{\partial u}{\partial y}(z, 0)
\end{aligned}
$$

(by Milne-Thomson rule replace x by z and y by 0.)

$$\therefore \qquad f(z) = \int \left[\frac{\partial u}{\partial x}(z, 0) - i\frac{\partial u}{\partial y}(z, 0)\right] dz + c$$

where c is an arbitrary constant of integration.

Separating the real & imaginary parts of $f(z)$ we can find the imaginary part $v(x, y)$.

■ **Note :**

1. The real part of f(z) should be identical to the given u(x,y).
2. If the imaginary part v(x, y) is given, then

   $$f(z) = \int \left[\frac{\partial v}{\partial y}(z, 0) + i\frac{\partial v}{\partial x}(z, 0)\right] dz + c$$

   Separating the real and imaginary parts of $f(z)$, we can find the real part u(x, y).

Example 4.7

Show that the function $f(z) = |z|^2$ is not analytic even though it is continuous every where and differentiable at zero.

Solution:

Let $f(z) = |z|^2 = z\overline{z}$.

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)\overline{(z + \Delta z)} - z\overline{z}}{\Delta z}$$

$$
\begin{aligned}
&= \lim_{\Delta z \to 0} \frac{\bar{z}\Delta z + \overline{z\Delta z} + \Delta z.\overline{\Delta z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left[\bar{z} + z\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right] \\
&= \lim_{r \to 0} \left[\bar{z} + z\frac{re^{-i\theta}}{re^{-i\theta}} + re^{-i\theta} \right] \quad (\Delta z = re^{i\theta}) \\
&= \overline{z} + ze^{-2i\theta}
\end{aligned}
$$

which does not tend to a unique limit as this limit depends on θ, the amplitude of Δz which is arbitrary.

$\therefore$ $f(z)$ is not differentiable at any point $z \neq 0$ and hence $f(z)$ is not an analytic at any point $z \neq 0$.

Thus continuity and differentiability of $f(z)$ at 0, is as below:

$|f(z) - f(0)| = |z|^2 = r^2 < \epsilon$ for all z for which $0 < |z - 0| < \epsilon$, where r is the modulus of z.

Thus $f(z)$ has the limit $f(0)$ as $z \to 0$ and hence $f(z)$ is continuous at 0.

$$
\begin{aligned}
\text{Also} \quad f'(0) &= \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\Delta z \Delta \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \Delta \bar{z} = 0.
\end{aligned}
$$

$$\text{Since} \quad z \to 0 \Rightarrow \overline{\Delta z} \to 0.$$

Though $f(z)$ is differentiable as the point 0, it is not analytic at $z = 0$. Thus $f(z)$ is not analytic.

Example 4.8

Show that $w = \log z$ is analytic in the complex plane except at the origin and that its derivative is $1/z$.

Solution:

$$
\begin{aligned}
\text{Let} \quad w &= u + iv = \log z = \log re^{i\theta} \\
&= \log r + i\theta \\
\therefore \quad u &= \log r \ \text{ and } \ v = \theta.
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence} \quad & \frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0 \\
& \frac{\partial v}{\partial r} = 0, \quad \frac{\partial v}{\partial \theta} = 1 \\
& \frac{\partial u}{\partial \theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -\frac{1}{r}\frac{\partial v}{\partial \theta}
\end{aligned}
$$

Thus the $C-R$ equations are satisfied for $r \neq 0$. since partial derivatives are continuous except at $z=0$, the given function is analytic except at $z=0$.

$$\begin{aligned}\frac{d}{dz}(\log z) &= e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\\ &= e^{-i\theta}\left(\frac{1}{r}\right) \quad \text{where } r\neq 0\\ &= \frac{1}{re^{i\theta}}=\frac{1}{z} \quad \text{for } z\neq 0.\end{aligned}$$

Example 4.9

Show that the function $\sqrt{|xy|}$ is not analytic at the origin, although $C-R$ equations are satisfied at the origin.

Solution:

$$\text{Let} \quad f(z)=u(x,y)+iv(x,y)=\sqrt{|xy|}$$

$$\therefore \quad u(x,y)=\sqrt{|xy|}; \quad v(x,y)=0.$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{\Delta x\to 0}\left[\frac{u(\Delta x,0)-u(0,0)}{\Delta x}\right]=0$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{\Delta y\to 0}\left[\frac{u(0,\Delta y)-u(0,0)}{\Delta y}\right]=0$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{\Delta x\to 0}\left[\frac{v(\Delta x,0)-v(0,0)}{\Delta x}\right]=0$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{\Delta y\to 0}\left[\frac{u(0,\Delta y)-v(0,0)}{\Delta y}\right]=0$$

$$\text{Hence} \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y} \quad \text{at the origin.}$$

$\therefore$ The C-R equations are satisfied at the origin.

Now

$$\begin{aligned}\lim_{\Delta z\to 0}\left[\frac{f(0+\Delta z)-f(0)}{\Delta z}\right] &= \lim_{\Delta z\to 0}\left[\frac{\sqrt{|\Delta x.\Delta y|-0}}{\Delta x+i\Delta y}\right]\\ &= \lim_{\Delta x\to 0}\left[\frac{\sqrt{|m.\Delta x^2|}}{\Delta x(1+im)}\right]\end{aligned}$$

$$(\text{Let } y = mx, \;\; \Delta y = m\Delta x)$$

$$= \frac{\sqrt{|m|}}{1 + im}$$

This limit is not unique, since it depends on m. Therefore its derivative does not exist at the origin. Hence $f(z)$ is not analytic at the origin.

Example 4.10

If $f(z)$ and $\overline{f(z)}$ are both analytic, show that $f(z)$ is a constant

Solution:

$$\begin{aligned} \text{Let} \quad f(z) &= u(x, y) + iv(x, y) \\ \therefore \quad \overline{f(z)} &= u(x, y) - iv(x, y) \\ &= u(x, y) + i(-v(x, y)) \end{aligned}$$

Since $f(z)$ is analytic, we have $u_x = v_y$ and $u_y = -v_x$
Since $\overline{f(z)}$ is analytic, we have $u_x = -v_y$ and $u_y = v_x$

Adding, we get,

$$\begin{aligned} u_x = 0 \;\text{ and }\; u_y &= 0 \\ \text{and hence } \; v_x = 0 \;\text{ and }\; v_y &= 0 \\ \therefore \quad f'(z) = u_x + iv_x &= 0 \end{aligned}$$

$\therefore \quad f(z)$ is a constant.

Example 4.11

If $f(z)$ is an analytic function whose real part is constant, prove that $f(z)$ is a constant function. **(AU 2011)**

Solution:

$$\text{Let } f(z) = u + iv \text{ which is analytic}$$

$$\text{Given} \quad u = C_1$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{-\partial v}{\partial x} = 0$$

$$\text{i.e., } \partial v / \partial x = 0$$

$$\text{Since } \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$$

$$v = \text{ constant } = C_2$$

$$\therefore \; f(z) = u + iv = C_1 + iC_2, \; a \text{ constant}$$

Example 4.12

Show that an analytic function in a region R, with constant modulus is constant.

(AU 2007)

Solution:

Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a region.

Since $|f(z)|$ is constant. Say $|f(z)| = c$.

$$\therefore \quad u^2 + v^2 = c$$

Differentiating partially with respect to x, we get

$$2uu_x + 2vv_x = 0$$

$$\text{or} \quad uu_x + vv_x = 0 \qquad (1)$$

$$\text{Similarly} \quad uu_y + vv_y = 0 \qquad (2)$$

$$\text{Using C-R equations} \quad uu_x - vv_y = 0$$

$$\text{and} \quad uu_x - vu_x = 0$$

Eliminating u_y from the above equation, we get

$$(u^2 + v^2)u_x = 0$$

$$\text{or} \quad u_x = 0 \quad (\text{because } u^2 + v^2 = c)$$

$$\text{Similarly} \quad v_x = 0$$

$$\therefore \quad f(z) = u_x + iv_x = 0$$

$\therefore \quad f(z)$ is a constant.

Example 4.13

Prove that every analytic function $w = u + iv$ can be expressed as a function of z alone, not as a function of $\overline{z}$. **(AU 2010, 2012)**

Solution:

Replace x and y by their equivalents

$$x = \frac{z + \overline{z}}{2} \qquad (i)$$

$$y = \frac{z - \overline{z}}{2i} \qquad (ii)$$

Then w can be considered to be a function of two new independent variables z and $\overline{z}$.

To prove that w depends on z alone and does not involve $\overline{z}$ it is enough to P.T $\dfrac{\partial w}{\partial \overline{z}} = 0$.

$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial (u+iv)}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} + i\frac{\partial v}{\partial \overline{z}}$$

$$= \left\{\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}}\right\} + i\left\{\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}}\right\}$$

From (i) & (ii) $\dfrac{\partial x}{\partial \overline{z}} = \dfrac{1}{2}$ and $\dfrac{\partial y}{\partial \overline{z}} = \dfrac{-1}{2i} = \dfrac{i^2}{2i} = \dfrac{i}{2}$

Hence $$\frac{\partial w}{\partial \overline{z}} = \left\{\frac{1}{2}\cdot\frac{\partial u}{\partial x} + \frac{i}{2}\cdot\frac{\partial u}{\partial y}\right\} + i\left\{\frac{1}{2}\cdot\frac{\partial v}{\partial x} + \frac{i}{2}\cdot\frac{\partial v}{\partial y}\right\}$$

(i.e.,) $$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left\{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right\} + \frac{i}{2}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}$$

Using C.R equations $\dfrac{\partial w}{\partial \overline{z}} = 0$

Hence there is no $\overline{z}$ in w.

Example 4.14

Test the analyticity of the function $f(x) = \overline{z}$. **(AU 2009)**

Solution:

$f(z)$	=	$\overline{z}$	z	=	$x+iy$
	=	$x-iy$	$\therefore\ \overline{z}$	=	$x-iy$
u	=	x	v	=	$-y$
u_x	=	1	v_x	=	0
u_y	=	0	v_y	=	-1

$\therefore u_x \neq v_y$ $\therefore f(z) = \overline{z}$ is not analytic.

Example 4.15

Prove that the following function are analytic and also find their derivatives using definition.

(i) $f(z) = z^2$ (ii) $f(z) = e^z$ (iii) $f(z) = \cos z$ (iv) $f(z) = \sinh z$ (v) $f(z) = z^n$

Solution:

(i)

$$\text{Let} \quad f(z) = u(x,y) + iv(x,y) = z^2$$
$$= (x+iy)^2$$
$$= (x^2 - y^2) + i(2xy)$$

$\therefore \quad u = x^2 - y^2$ and $v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y$$
$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$
$$\text{Clearly} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$\therefore$ The four partial derivatives exist, they are continuous and $C - R$ equations are satisfied for all finite values of x and y.

$\therefore \quad f(z)$ is analytic everywhere.

$$\text{Now} \quad f(z) = u_x + iv_x$$
$$= 2x + i2y$$
$$= 2(x+iy)$$
$$= 2z$$

(ii) Let $f(z) = u + iv = e^z = e^{x+iy}$

$$= e^x \cdot e^{iy}$$
$$= e^x(\cos y + i \sin y)$$

$u = e^x \cos y$ and $v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial x} = e^x \sin y$$
$$\frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial v}{\partial x} = e^x \cos y$$
$$\text{Clearly} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

All the four partial derivatives exist everywhere and are continuous and satisfy the $C - R$ equations.

$\therefore \quad f(z)$ is analytic everywhere.

$$\text{Now} \quad f(z) = u_x + iv_x$$
$$= e^x \cos y + ie^x \sin y$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x e^{iy} = e^{x+iy} = e^z$$

(iii) Let $f(z) = u + iv = \cos z$

$$= \cos(x + iy)$$
$$= \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cos hy - i \sin x \sin hy$$

$u = \cos x \cos hy$ and $v = -\sin x \sin hy$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial v}{\partial x} = -\cos x \sinh y$$
$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial v}{\partial y} = -\sin x \cosh y$$
$$\text{Clearly} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

The four partial derivatives exist and are continuous and also satisfy the $C - R$ equation.

$\therefore f(z)$ is analytic.

$$\text{Now} \quad f'(z) = u_x + iv_x$$
$$= -\sin x \cos hy - i \cos x \sin hy$$
$$= -(\sin x \cos iy + \cos x \sin iy)$$
$$= -(\sin(x + iy))$$
$$= -\sin z.$$

(iv) Let $f(z) = u + iv = \sin hz$

$$= \sin h(x + iy)$$
$$= \sin hx \cos y + i \cos hx \sin y$$

$u = \sin hx \cos y$ and $v = \cos hx \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial v}{\partial x} = \sinh x \sin y$$
$$\frac{\partial u}{\partial y} = -\cosh x \sin y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y$$
$$\text{Clearly} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The four partial derivatives exists and are continuous and the $C - R$ equations are satisfied.

$\therefore f(z)$ is analytic

$$\text{Now} \quad f'(z) = u_x + iv_x$$
$$= \cos hx \cos y + i \sin hx \sin y$$
$$= \cos h(x+iy)$$
$$= \cos hz$$

(v) Let $f(z) = u(r,\theta) + iv(r,\theta) = z^n = (re^{i\theta})^n$

$$= r^n(\cos n\theta + i \sin n\theta)$$

$u = r^n \cos n\theta$ and $v = r^n \sin n\theta$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \quad \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$
$$\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta, \quad \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$
$$\text{Clearly} \quad \frac{\partial u}{\partial \theta} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \text{and} \quad \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The partial derivatives exists and are continuous and the $C - R$ equations are satisfied.

$\therefore f(z)$ is analytic

$$\text{Now} \quad f'(z) = e^{-i\theta}(u_r + iv_r)$$
$$= e^{-i\theta} nr^{n-1}(\cos n\theta + i \sin n\theta)$$
$$= nr^{n-1}e^{-i}e^{in\theta}$$
$$= nr^{n-1}e^{i(n-1)\theta}$$
$$= n(re^{i\theta})^{n-1}$$
$$= nz^{n-1}$$

Example 4.16

Find where each of the following functions ceases to be analytic

(i) $\tan^2 z$ (ii) $\dfrac{z}{z^2-1}$ (iii) $\dfrac{z+i}{(z-i)^2}$ (iv) $\sin\ h^{-1}z$

Solution:

(i) Let $f(z) = \tan^2 z$

then $f'(z) = 2\tan z \sec^2 z = \dfrac{2\sin z}{\cos^3 z}$

when $\cos^3 z = 0$ $f'(z) \to \infty$

i.e. when $z = \dfrac{(2n-1)\pi}{2}$

$\therefore$ $f(z)$ is not analytic at $z = \dfrac{(2n-1)\pi}{2}, n = 1, 2, 3\cdots.$

(ii) $f(z) = \dfrac{z}{z^2-1}$

$$f(z) = \frac{(z^2-1)(1)-(z)(2z)}{(z^2-1)^2} = -\frac{(z^2+1)}{(z^2-1)^2}$$

$f'(z)$ does not exist i.e. if $(z^2-1)^2 = 0$ then $f'(z) \to \infty$
i.e. if $z = \pm 1$ $f'(z) \to \infty$

$\therefore$ $f(z)$ is not analytic at the points $z = \pm 1$.

(iii) Let $f(z) = \dfrac{z+i}{(z-i)^2}$

$$f'(z) = \frac{(z-i)^2(1)-(z+i)2(z-i)}{(z-i)^2} = -\frac{z+3i}{(z-i)^3}$$

$f'(z)$ does not exist. i.e. if $(z-i)^3 = 0$ then $f'(z) \to \infty$
Hence $f(z)$ is not analytic at the point $z = i$.

(iv) $f(z) = \sinh^{-1} z$

$f'(z) = \dfrac{1}{\sqrt{z^2+1}}, f'(z) \to \infty$ when $z^2+1 = 0$ i.e. $z = \pm i$
Hence $f(z)$ is not analytic at the points $z = \pm i$.

Example 4.17

Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic. **(AU 2010)**

Solution:

Given $f(z) = (x + ay) + i(bx + cy) = u + iv$

Take $u = x + ay$; $\quad v = bx + cy$.

If the function is an analytic then C.R function is satisfied.

$u_x = +v_y$; $\qquad u_y = -v_x$

$u_x = 1 \quad ; \quad u_y = a \quad ; \quad v_x = b \quad ; \quad v_y = c$

$\boxed{c = 1} \qquad \boxed{a = -b}$

Hence $a = -b$ and $c = 1$.

Example 4.18

Is $f(z) = z^3$ analytic? **(AU 2009)**

Solution:

$$f(z) = z^3$$

$$\begin{aligned} u + iv &= (x + iy)^3 \\ &= x^3 - iy^3 + i3x^2y - 3xy^2 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3) \end{aligned}$$

$$\begin{aligned} u &= x^3 - 3xy^2 & v &= 3x^2y - y^3 \\ u_x &= 3x^2 - 3y^2 & v_x &= 6xy \\ u_y &= -6xy & v_y &= 3x^2 - 3y^2 \end{aligned}$$

$\because \quad u_x = v_y$ and $u_y = -v_x$

$\therefore \quad f(z)$ is analytic.

Example 4.19

Prove that $z\bar{z}$ is not analytic. **(AU 2009)**

Solution:

$$f(z) = z\bar{z}$$

$$f(z) = (x + iy)(x - iy) = x^2 + y^2$$

Here $\quad u(x, y) = x^2 + y^2, \quad v(x, y) = 0$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$$

The Cauchy - Rieman equation are satisfied if $2x = 0$ and $2y = 0$.

(i.e.,) $x = 0$ and $y = 0$.

Thus, the C.R equation are satisfied at the origin; further the partial derivatives of u and v are continuous. Hence $z\overline{z}$ has a derivatives at the origin (only).

Though $f(z)$ has a derivative at the origin, there is no neighbourhood of the origin where it has a derivative.

$\therefore$ $f(z)$ is not analytic.

Example 4.20

Find the critical points of the mapping $\omega = z^2$. **(AU 2009)**

Solution:

$$\omega = z^2 \qquad (1)$$

$$\frac{d\omega}{dz} = 2z \qquad (2)$$

Critical points occur at $\frac{d\omega}{dz} = 0$. Hence from (2) at $2z = 0$.

(i.e.,) $z = 0$

From (2),

$$\frac{1}{2z} = \frac{dz}{d\omega} \qquad (3)$$

Critical points occur also at $\frac{dz}{d\omega} = 0$

Hence from (3) critical points occur at $\omega = 0$ also

$$\text{(i.e.,)} \quad \text{at } z^2 = 0$$
$$\text{(i.e.,)} \quad \text{at } z = 0$$

Example 4.21

If $f(z) = u + iv$ is analytic prove that $u - iv$ and $u + iv$ are not analytic.

Solution:

Let $f(z) = v + iu$ one not analytic

$\therefore$ $u_x,\ u_y,\ v_x$ & v_y exist and satisfy the C-R equations namely

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial y}{\partial y} = -\frac{\partial v}{\partial x}$$

(i) Now let $g(z) = U + iV = u - iv$

then $\quad U_x = u_x \qquad V_x = -v_x = u_y$

$\quad U_y = +u_y \qquad V_y = -v_y = -u_x$

The C-R equations do not hold good

$\therefore \quad g(z) = u - iv$ is not analytic.

(ii) Now let $g(z) = U + iV = v + iu$

$$\text{then} \quad \frac{\partial U}{\partial x} = \frac{\partial v}{\partial x} \qquad \frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial U}{\partial y} = \frac{\partial v}{\partial y} \qquad \frac{\partial V}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the C-R equations do not hold good.

$\therefore \quad g(z) = v + iu$ is not analytic.

Example 4.22

Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

(AU 2010)

Solution:

$$u_x = 3x^2 - 3y^2 + 6x \quad ; \quad u_y = -6xy - 6y$$
$$u_{xx} = 6x + 6 \quad ; \quad u_{yy} = -6x - 6$$

Here $u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$. The Laplace equation is satisfied. Hence u is harmonic.

Example 4.23

Show that $\varphi = 3x^2y - y^3$ is harmonic. **(AU 2009, 2010)**

Solution:

$$\varphi = 3x^2y - y^3$$

$$\varphi_x = 6xy \qquad \varphi_y = 3x^2 - 3y^2$$
$$\varphi_{xx} = 6y \qquad \varphi_{yy} = -6y$$

$\therefore \quad \varphi_{xx} + \varphi_{yy} = 6y - 6y = 0 \qquad \therefore \varphi$ is a harmonic function.

Example 4.24

Show that the function $u(x, y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find the harmonic conjugate function $v(x, y)$ such that $u + iv$ is an analytic function.

(AU 2009)

Solution:

Given $u(x,y) = 3x^2y + x^2 - y^3 - y^2$

$$\frac{\partial u}{\partial x} = 6xy + 2x \qquad \frac{\partial^2 u}{\partial x^2} = 6y + 2$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 2y \qquad \frac{\partial^2 u}{\partial y^2} = -6y - 2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and so u is a harmonic function

Since v is the conjugate harmonic of u, $u + iv$ is analytic.

$$\text{Now} \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \frac{-\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= (-3x^2 + 3y^2 + 2y)dx + (6xy + 2x)dy$$

$$\therefore \; v = \int [-3x^2 + 3y^2 + 2y)dx + (6xy + 2x)dy + c$$

$$= \int (M\,dx + N\,dy) + c$$

$$= \int M\,dx \text{ (keeping } y \text{ constant)} + \int \text{(terms independent of } x \text{ in } N)\,dy$$

$$= \int (-3x^2 + 3y^2 + 2y)dx + c$$

$$v = \frac{-\cancel{3}x^3}{\cancel{3}} + 3y^2x + 2yx + c,$$ which is the harmonic conjugate of u

$$\therefore \; f(z) = u + iv$$

$$= (3x^2y + x^2 - y^3 - y^2) + i(3y^2x + 2xy - x^3) + c$$

$$= -i[x^3 + 3x^2iy + 3i^2y^2 + i^3y^3] + [x^2 + 2xiy + i^2\,y^2] + c$$

$$= -i(x+iy)^3 + (x+iy)^2 + c$$

$$f(z) = -iz^3 + z^2 + c$$

Example 4.25

Prove that $u = 2x - x^3 + 3xy^2$ is harmonic and determine its harmonic conjugate. **(AU 2010)**

Solution:

$$u = 2x - x^3 + 3xy^2$$

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2, \qquad \frac{\partial u}{\partial y} = 0 + 6xy$$

$$\frac{\partial^2 u}{\partial x^2} = -6x, \qquad \frac{\partial^2 u}{\partial y^2} = 6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$

$$\Rightarrow \quad u \text{ is harmonic}$$

$$f(z) = \int \frac{\partial u}{\partial x}(z,\ 0)\, dz - i \int \frac{\partial u}{\partial y}(z,\ 0)\, dz$$

$$= \int (2 - 3z^2)\, dz - i \int 0\, dz$$

$$f(z) = 2z - z^3 \qquad \text{Put } z = x + iy$$

$$= 2(x + iy) - (x + iy)^3$$

$$= 2x + 2iy - x^3 + xy^2 - 2ix^2y - ix^2y + iy^3 + 2xy^2$$

$$= 2x - x^3 + 3xy^2 + i\left(2y + y^3 - 3x^2y\right)$$

$$= u + iv$$

$$\therefore \quad v = 2y + y^3 - 3x^2y$$

Example 4.26

Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic and its harmonic conjugate function. **(AU 2009)**

Solution:

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2$$

$$u_x = 3x^2 - 3y^2 + 6x, \qquad u_{xx} = 6x + 6$$

$$u_y = -6xy - 6y \qquad u_{yy} = -6x - 6$$

$u_{xx} + u_{yy} = 0$ and so u is a harmonic function.

$\because$ v is the conjugate harmonic of u, $u + iv$ is analytic.

$\therefore$ By C.R equations, $u_x = v_y$ and $u_3 = -v_x$

$$\text{Now} \qquad dv = \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy = \frac{-\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy$$

$$dv = (6xy + 6y)dx + (3x^2 - 3y^2 + 6x)dy$$

$$\text{Integral,} \qquad v = \int \left[(6xy + 6y)dx + (3x^2 - 3y^2 + 6x)dy\right]$$

$$= \frac{6x^2y}{2} + 6xy - \frac{3y^3}{3} + 3x^2y + 6xy = 3x^2y + 6xy - y^3 + C$$

$$\text{Let} \quad w = u + iv = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y + 6xy - y^3 + C)$$

$$= z^3 + 3z^2 + 1 + iC, \qquad \text{by Milne-Thomson rule}$$

Example 4.27

Show that the following functions are harmonic and find the corresponding *conjugate harmonic functions*

(i) $u = e^x \cos y$ (ii) $v = \log(x^2 + y^2)$ **(AU 2008)**

Solution:

Given $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \therefore \quad u(x, y) \text{ is a harmonic function}$$

But $f(z) = u + iv$ where $u = e^x \cos y$

$$\therefore \quad f'(z) = \frac{\partial u(x,y)}{\partial x} - i\frac{\partial u(x,y)}{\partial y} \qquad \text{(by C-R equations)}$$

$$= \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y} \qquad \text{(by Milne - Thomson rule)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial u(z,0)}{\partial x} - \frac{i\partial u(z,0)}{\partial y}\right] dz + c$$

$$= \int e^z - i(0)dz + c$$

$$= \int e^z dz + c$$

$$= e^z + c$$

Now $\quad f(z) = e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$

$\therefore \quad v(x,y) = e^x \sin y$

(ii) Given $v(x,y) = \log(x^2+y^2)$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} \qquad \frac{\partial^2 v}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0$$

$\therefore \quad v(x,y)$ is a harmonic function.

Let $\quad f(z) = u + iv$ where $v = \log(x^2+y^2)$

$$\therefore \quad f'(z) = \frac{\partial v}{\partial y} + \frac{i\partial v}{\partial x}$$

$$= \frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x} \qquad \text{(by Milne - Thomson rule)}$$

$$\text{or} \quad f(z) = \int \left[\frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x}\right] dz + c$$

$$= \int (0 + i2/z)dz + c$$

$$= i2\log z + c$$

Now $\quad f(z) = 2i\log z$

$$= 2i\log(re^{i\theta})$$

$$
\begin{aligned}
&= 2i(\log r + i\theta)\\
&= 2i\log r - 2\theta\\
&= 2i\log(x^2+y^2)^{1/2} - 2\tan^{-1} y/x\\
&= i\log(x^2+y^2) - 2\tan^{-1} y/x\\
\therefore \quad u(x,y) &= -2\tan^{-1} y/x
\end{aligned}
$$

Example 4.28

Find the regular function whose imaginary part $e^x \sin y$. **(AU 2009)**

Solution:

$$
\begin{aligned}
v &= e^x \sin y\\
\frac{\partial v}{\partial y} &= e^x \cos y = \phi_1(x,\ y)\\
\therefore \quad \phi_1(z,\ 0) &= e^z\\
\frac{\partial v}{\partial x} &= e^x \sin y = \phi_2(x,\ y)\\
\therefore \quad \phi_2(z,\ 0) &= 0\\
\therefore \quad f(z) &= \int \phi_1(z,\ 0)\, dz + \int \phi_2(z,\ 0)\, dz\\
&= \int e^z\, dz + 0\\
&= e^z + C
\end{aligned}
$$

Example 4.29

Show that $\dfrac{x}{x^2+y^2}$ is harmonic. **(AU 2009)**

Solution:

$$
\begin{aligned}
\text{Let} \quad f &= \frac{x}{x^2+y^2}\\
\frac{\partial f}{\partial x} &= \frac{y^2-x^2}{(x^2+y^2)^2}\\
\frac{\partial^2 f}{\partial x^2} &= \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)(2(x^2+y^2)(2x))}{(x^2+y^2)^4}\\
&= \frac{2x^3-6xy^2}{(x^2+y^2)^3}
\end{aligned}
$$

$$\frac{\partial f}{\partial y} = \frac{-x}{(x^2+y^2)^2}\,(2y)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)^2(-2x)+(2xy)\cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = \frac{2x^3-6xy^2-2x^3+6xy^2}{(x^2+y^2)^3} = \frac{0}{(x^2+y^2)^3} = 0$$

$\therefore$ f is harmonic.

■ **Note:**

(i) When the real part u is given use the formula $f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

(ii) Where the imaginary part v is given use the formula $f'(z) = \dfrac{\partial v}{\partial y} + i\dfrac{\partial v}{\partial x}$

Example 4.30

Find the analytic function whose real part is $\dfrac{x}{x^2+y^2}$. **(AU 2009)**

Solution:

Let $f(z) = u(x,y) + iv(n,y)$

Given $u = \dfrac{x}{x^2+y^2}$

$$\therefore \quad \frac{\partial u}{\partial x}(x,y) = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u(z,0)}{\partial x} = -\frac{1}{z^2} \qquad (1)$$

$$\therefore \quad \frac{\partial u(x,y)}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial u(z,0)}{\partial y} = 0 \qquad (2)$$

$$\text{Now} \quad f'(z) = \frac{\partial u(x,y)}{\partial x} - i\frac{\partial u(x,y)}{\partial y}$$
$$= \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}$$
$$\therefore \quad f(z) = \int \left[\frac{\partial u(z,0)}{\partial x} - i\frac{\partial y(z,0)}{\partial y}\right] dz + c$$
$$= \int -\frac{1}{z^2} dz + c = \frac{1}{z} + c$$
$$\therefore \quad f(z) = \frac{1}{z} + c.$$

Example 4.31

Find the analytic function $w = u + iv$ if $u = e^x(x\cos y - y\sin y)$. Hence find the harmonic conjugate v. **(AU 2007, 2009, 2010)**

Solution:

Given $u(x,y) = e^x(x\cos y - y\sin y)$

$$\frac{\partial u(x,y)}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$$
$$= e^x(x\cos y + \cos y - y\sin y)$$
$$\therefore \quad \frac{\partial u(z,0)}{\partial x} = e^z(z+1) \tag{1}$$
$$\frac{\partial u}{\partial y}(x,y) = e^x(-x\sin y - (\sin y + y\cos y))$$
$$= e^x(-x\sin y - \sin y - y\cos y)$$
$$\therefore \quad \frac{\partial u}{\partial y}(z,0) = e^z(0) = 0 \tag{2}$$
$$\frac{dw}{dz} = \frac{\partial u(x,y)}{\partial x} - i\frac{\partial u(x,y)}{\partial y}$$
$$= \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y} \qquad \text{(by Milne-Thomson rule)}$$
$$\therefore \quad w = \int \left[\frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}\right] dz + c$$
$$= \int (1+z)e^z\, dz + c = ze^z + c, \qquad \text{using Bernoulli's formula which is given below}$$
$$\therefore \quad w = ze^z + c$$

$$\text{Now} \quad w = (x+iy)e^{x+iy}$$
$$= (x+iy)(e^x \cdot e^{iy})$$
$$= e^x(x+iy)(\cos y + i\sin y)$$
$$= e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y)$$
$$\therefore \quad v(x,y) = e^x(y\cos y + x\sin y) + c$$

■ **Note:** Bernoulli's formula $\int PQdx = (P)(Q_1) - (P')(Q_2) + (P'')(Q_3)\cdots$

e.g $\int (1+z)e^z dz = (1+z)(e^z) - (1)(e^z) = e^z + ze^z - e^z = ze^z$

Where dashes $\rightarrow$ differentiation
suffix $\rightarrow$ integration

Example 4.32

Find the analytic function $f(z) = u + iv$ if $v = e^{2x}(x\cos 2y - y\sin 2y)$. Hence find u. **(AU 2011)**

Solution:

Given $v = e^{2x}(x\cos 2y - y\sin 2y)$

$$\therefore \quad \frac{\partial v}{\partial x}(x,y) = e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial v}{\partial x}(z,0) = e^{2z}z + 2e^{2z}z(z) = e^{2z}z(2z+1) \qquad (1)$$

$$\frac{\partial v}{\partial y}(x,y) = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

$$\frac{\partial v}{\partial y}(z,0) = e^{2z}(0) = 0 \qquad (2)$$

$$\text{Now} \quad f'(z) = \frac{\partial v(x,y)}{\partial y} + i\frac{\partial v(x,y)}{\partial x}$$

$$= \frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x} \qquad \text{(by Milne-Thomson rule)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x}\right] dz + c$$

$$= \int i \cdot e^{2z}(2z+1)dz + c$$
$$= ize^{2z} + c$$
$$\therefore \quad f(z) = ize^{2z} + c$$
$$\text{Now} \quad f(z) = i(x+iy)e^{2(x+iy)} + c = (ix-y)(e^{2x}e^{2iy}) + c$$
$$= e^{2x}(ix-y)(\cos 2y + i\sin 2y) + c$$
$$= -e^{2x}(x\sin 2y + y\cos 2y) + ie^{2x}(x\cos 2y - y\sin 2y) + c$$
$$\therefore \quad u = -e^{2x}(x\sin 2y + y\cos 2y) + c$$

■ **Note:**

$$\int e^{2z}(2z+1)dz = \left[(2z+1)\left(\frac{e^{2z}}{2}\right)\right] - \left[(2)\left(\frac{e^{2z}}{4}\right)\right]$$
$$= ze^{2z} + \frac{e^{2z}}{2} - \frac{e^{2z}}{2} = ze^{2z}$$

Example 4.33

Find the analytic function $f(z) = u + iv$, given that $v = e^{x^2-y^2}\cos 2xy$. Hence find u, the harmonic conjugate of v.

Solution:

Given $v = e^{x^2-y^2}\cos 2xy$

$$\frac{\partial v}{\partial x}(x,y) = e^{x^2-y^2}(-2y\sin 2xy) + 2x\, e^{x^2-y^2}\cos 2xy$$
$$\frac{\partial v(z,0)}{\partial x} = 2ze^{z^2}$$
$$\frac{\partial v(x,y)}{\partial y} = e^{x^2-y^2}(-2x\sin 2xy) - 2ye^{x^2-y^2}\cos 2xy$$
$$\therefore \quad \frac{\partial v(z,0)}{\partial y} = 0$$
$$f'(z) = \frac{dw}{dz} = \frac{\partial v(z,y)}{\partial y} + i\frac{\partial v(x,y)}{\partial x}$$
$$\therefore \quad f(z) = \int \left[\frac{\partial v}{\partial y}(z,0) + i\frac{\partial v}{\partial x}(z,0)\right]dz + c$$
$$= \int 2ize^{z^2}dz + c \qquad \text{put } z^2 = t \qquad 2zdz = dt$$

$$= i\int e^t dt + c = ie^t + c$$

$$\therefore \quad f(z) = ie^{z^2} + c$$

$$\text{Now} \quad f(z) = ie^{(x+iy)^2}$$

$$= ie^{(x^2-y^2+2ixy)} + c$$

$$= i[e^{x^2-y^2}e^{2ixy} + c]$$

$$= i\left[e^{x^2-y^2}(\cos 2xy + i\sin 2xy) + c\right]$$

$$= ie^{x^2-y^2}\cos 2xy - e^{x^2-y^2}\sin 2xy + c$$

$$\therefore \quad u(x,y) = -e^{x^2-y^2}\sin 2xy + c$$

Example 4.34

If $u(x,y) = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]$, find the analytic function. $f(z) = u + iv$ **(AU 2007, 2009)**

Solution:

Given $u(x,y) = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]$

$$\therefore \quad \frac{\partial u(x,y)}{\partial x} = -e^{-x}[(x^2-y^2)\cos y + 2xy\sin y] + e^{-x}[2x\cos y + 2y\sin y]$$

$$\frac{\partial u(z,0)}{\partial x} = -e^{-z}[z^2] + e^{-z}(2z) \qquad (1)$$

$$\frac{\partial u(x,y)}{\partial y} = -e^{-x}[-(x^2-y^2)\sin y - 2y\cos y + 2x\sin y + 2xy\cos y]$$

$$\frac{\partial u(z,0)}{\partial y} = 0 \qquad (2)$$

$$\text{now} \quad f'(z) = \frac{\partial u(x,y)}{\partial x} - i\frac{\partial v(x,y)}{\partial x}$$

$$= \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}$$

$$\therefore \quad f(z) = \int\left[\frac{\partial u(z,0)}{\partial x} - i\frac{\partial u}{\partial y}(z,0)\right]dz + c \qquad \text{from (1) and (2)} \qquad (3)$$

$$= \int e^{-z}(2z - z^2)dz + c$$
$$= \int e^{-z}2zdz - \int e^{-z}z^2dz$$
$$= 2\int e^{-z}zdz - \left[(z^2)e^{-z} - \int \frac{e^{-z}}{-1}2zdz\right]$$
$$= 2\int e^{-z}zdz + z^2e^{-z} - 2\int e^{-z}zdz$$
$$= e^{-z}z^2 + c$$
$$\therefore \quad f(z) = e^{-z}z^2 + c$$

Example 4.35

Construct the analytic function whose imaginary part is $e^{-x}(x\cos y + \sin y)$ and which equal 1 at the origin. **(AU 2009)**

Solution:

Given $v = e^{-x}(x\cos y + \sin y)$ and $f(0) = 1$

$$\frac{\partial v(x,y)}{\partial x} = -e^{-x}(x\cos y + \sin y) + e^{-x}(\cos y)$$
$$\frac{\partial v(z,0)}{\partial x} = e^{-z}(1-z) \qquad [\text{put } x = z,\ y = 0] \qquad (1)$$
$$\frac{\partial v(x,y)}{\partial y} = e^{-x}(-x\sin y + \cos y)$$
$$\frac{\partial v(z,0)}{\partial y} = e^{-z} \qquad (2)$$
$$f'(z) = \frac{\partial v(x,y)}{\partial y} + i\frac{\partial v(x,y)}{\partial x}$$
$$= \frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x} \qquad \text{(by Milne - Thomson rule)}$$
$$\therefore \quad f(z) = \int \left[\frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x}\right] dz + c$$
$$= \int e^{-z} + i(e^{-z}(1-z))dz + c$$
$$= -e^{-z} + i(ze^{-z}) + c$$

Also given $f(0) = 1$

$$\therefore \quad c - 1 = 1$$
$$\text{or} \quad c = 2$$
$$\therefore \quad f(z) = -e^{-z} + i\,ze^{-z} + 2$$

Example 4.36

If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find the corresponding analytic function $f(z) = u + iv$. Hence find, the harmonic conjugate of v. **(AU 2008)**

Solution:

Given $\quad u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$

$$\frac{\partial u(x,y)}{\partial x} = \frac{2\cos 2x(\cosh 2y + \cos 2x) - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y + 2}{(\cosh 2y + \cos 2x)^2} \qquad \text{put } x = z \text{ and } y = 0$$

$$\frac{\partial u(z,0)}{\partial x} = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} = 2.\frac{1}{1 + \cos 2z}$$

$$= \frac{2}{2\cos^2 z} = \sec^2 z \tag{1}$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2} \qquad \text{put } x = z,\ y = 0$$

$$\frac{\partial u(z,0)}{\partial y} = 0 \tag{2}$$

Now $\quad f'(z) = \dfrac{\partial u(x,y)}{\partial x} - i\dfrac{\partial u(x,y)}{\partial y}$

$$= \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y} \qquad \text{(by Miline - Thomson rule)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}\right] dz + c$$

$$= \int \sec^2 z\, dz + c$$

$$f(z) = \tan z + c$$

Now $\quad f(z) = u + iv = \tan z + c$

$$= \tan(x + iy) + c$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} + c$$

$$= \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} + c$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} + c$$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\therefore \quad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Example 4.37

Find the analytic function, whose real part is $\frac{\sin 2x}{(\cos h\ 2y - \cos 2x)}$. **(AU 2009)**

Solution:

$$u = \frac{\sin 2x}{\cos h\ 2y - \cos 2x}$$

$$\phi_1(x,\ y) = u_x$$

$$= \frac{(\cos h\ 2y - \cos 2x)2\ \cos 2x - \sin 2x(0 + 2\ \sin 2x)}{(\cos h\ 2y - \cos 2x)^2}$$

$$= \frac{2(\cos 2x\ \ \cos h\ 2y - 1)}{(\cos h\ 2y - \cos 2x)^2}$$

$$\phi_1(z,\ 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= \frac{2}{\cos 2z - 1}$$

$$\phi_2(x,\ y) = u_y$$

$$= \frac{(\cos h\ 2y - \cos 2x)0 - 2\ \sin 2x\ \ \sin h\ 2y}{(\cos h\ 2y - \cos 2x)^2}$$

$$= \frac{-2\ \sin 2x\ \ \sin h\ 2y}{(\cos h\ 2y - \cos 2x)^2}$$

$$\phi_2(z,\ 0) = \frac{0}{Dr} = 0$$

By Milne's Thomson method,

$$f'(z) = \phi_1(z,\ 0) - i\ \phi_2(z,\ 0)$$

$$= \frac{2}{\cos 2z - 1}$$

$$= -\frac{2}{1-\cos 2z}$$
$$= -\frac{2}{2\ \sin^2 z}$$
$$= -\frac{1}{\sin^2 z}$$
$$= -\text{cosec}^2\ z$$
$$\therefore \qquad f(z) = -\int \text{cosec}^2\ z\,dz$$
$$= \cot z + C$$

Example 4.38

Find the imaginary part of an analytic function whose real part is $u = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$. **(AU 2009)**

Solution:

Given $u = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$

$$\frac{\partial u(x,y)}{\partial x} = \frac{4\cos 2x(e^{2y}+e^{-2y}-2\cos 2x) - 2\sin 2x(4\sin 2x)}{(e^{2y}+e^{-2y}-2\cos 2x)^2}$$
$$\frac{\partial u(z,0)}{\partial x} = \frac{(1+1-2\cos 2z)4\cos 2z - 8\sin^2 2z}{(1+1-2\cos 2z)^2}$$
$$= \frac{8\cos 2z - 8\cos^2 2z - 8\sin^2 2z}{(2-2\cos^2 2z)^2}]$$
$$= \frac{-8(1-\cos 2z)}{4(1-\cos 2z)^2} = \frac{-2}{2\sin^2 z} = -\text{cosec}^2 z \qquad (1)$$
$$\frac{\partial u(x,y)}{\partial y} = \frac{-2\sin 2x(2e^{2y}-2e^{-2y})}{(e^{2y}+e^{-2y}-2\cos 2z)^2}$$
$$\frac{\partial u(z,0)}{\partial y} = \frac{-4\sin 2z + 4\sin 2z}{(1-2\cos 2z)^2} = 0 \qquad (2)$$

Now $\quad f'(z) = \dfrac{\partial u(x,y)}{\partial x} - i\dfrac{\partial u(x,y)}{\partial y}$

$$= \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y} \quad \text{(by Miline - Thomson rule)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial y(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}\right] dz + c$$

$$= \int -\text{cosec}^2 z \, dz + c$$

$$f(z) = \cot z + c \qquad \text{(from (1) and (2))}$$

Now

$$f(z) = \cot(x+iy) + c$$

$$= \frac{\cos(x+iy)}{\sin(x+iy)} + c$$

$$= \frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)} + c$$

$$= \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x} + c$$

$$= \frac{\sin 2x - i\sinh 2y}{\frac{e^{2y}+e^{-2y}}{2} - \cos 2x} + c$$

$$= \frac{2\sin 2x - i2\sinh 2y}{e^{2y}+e^{-2y} - 2\cos 2x} = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x} + i\frac{-2\sinh 2y}{e^{2y}+e^{-2y}-2\cos 2x}$$

$$\therefore \quad v(x,y) = \frac{-2\sinh 2y}{e^{2y}+e^{-2y}-2\cos 2x}$$

Example 4.39

Find the imaginary part of the analytic function whose real part is $\dfrac{2\cos x\cosh y}{\cos 2x + \cosh 2y}$.

(AU 2010)

Solution:

Given $u = \dfrac{2\cos x\cosh y}{\cos 2x + \cosh 2y}$

$$\therefore \frac{\partial u(x,y)}{\partial x} = \frac{(-2\sin x\cosh y)(\cos 2x + \cosh 2y) - (2\cos x\cosh y)(-2\sin 2x)}{(\cos 2x + \cosh 2y)^2}$$

$$\frac{\partial u(z,0)}{\partial x} = \frac{(\cos 2z + 1)(-2\sin z) - (2\cos z)(-2\sin 2z)}{(1+\cos 2z)^2}$$

$$= \frac{-4\cos^2 z\sin z + 8\cos^2 z\sin z}{(2\cos^2 z)^2}$$

$$= \frac{\sin z}{\cos z\cos z} = \tan z\sec z \qquad (1)$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{(\cos 2x - \cosh 2y)(2\cos x \sinh y) – (2\cos x \cosh y) – (2\sinh y)}{(\cos 2x + \cosh 2y)^2}$$

$$\frac{\partial u(z,0)}{\partial y} = 0 \qquad (2)$$

$$\text{Now} \quad f'(z) = \frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y} = \frac{\partial u(x,y)}{\partial x} - i\frac{\partial u(x,y)}{\partial y}$$

(by Milne - Thomson rule)

$$\therefore \quad f(z) = \int \left[\frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}\right] dz + c$$

$$= \int \sec z \ \tan z \ dz + c \qquad \text{(from (1) and (2))}$$

$$f(z) = \sec z + c$$

$$\therefore \quad f(z) = \sec(x + iy) + c$$

$$= \frac{1}{\cos(x+iy)} + c$$

$$= \frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} + c$$

$$= \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y} + i\frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

$$\therefore \quad v(x,y) = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

Example 4.40

Construct the analytic function $f(z)$ whose imaginary part is $e^{-2x}(y\cos 2y - x\sin 2y)$.

Solution:

Given $v(x,y) = e^{-2x}(y\cos 2y - x\sin 2y)$

$$\therefore \quad \frac{\partial v}{\partial x}(x,y) = e^{-2x}(-\sin y)$$

$$\frac{\partial v}{\partial x}(z,0) = 0 \qquad (1)$$

$$\frac{\partial v}{\partial x}(x,y) = e^{-2x}\left[-2y\sin 2y + \cos 2y - 2x\cos 2y\right]$$

$$\frac{\partial v}{\partial y}(z,0) = e^{-2z}(-2z+1) \qquad (2)$$

$$\text{Now} \quad f'(z) = \frac{\partial v}{\partial y}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

$$= \frac{\partial v}{\partial y}(z,0) + i\frac{\partial v}{\partial x}(z,0) \quad \text{(by Milne-Thomson method)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial v}{\partial y}(z,0) + i\frac{\partial v}{\partial x}(z,0)\right] dz + c$$

$$= \int \left[e^{-2z}(-2z+1)\right] dz + c \quad \text{(from (1) and (2))}$$

$$= ze^{-2z} - \int e^{-2z} dz + \int e^{-2z} dz + c$$

$$= ze^{-2z} + c$$

$$f(z) = (x+iy)e^{-2(x+iy)} + c$$

$$= (x+iy).e^{-2x}.e^{-i2y}$$

$$= e^{-2x}(x+iy)(\cos 2y - i\sin 2y)$$

$$= e^{-2}(x\cos 2y - ix\sin 2y + iy\cos 2y + y\sin 2y)$$

$$= e^{-2x}(x\cos 2y + y\sin 2y) + ie^{-2x}(y\cos 2y - x\sin 2y)$$

$$\therefore \quad u(x,y) = e^{-2x}(x\cos\ 2y + y\sin\ 2y)$$

■ **Note:** Let $w = f(z) = u + iv$ be an analytic function.

(i) If $u + v$ is given, then let $F(z) = U + iV$ is a new analytic function whose real part $U = u + v$, then the required analytic function $f(z)$ is

$$f(z) = \frac{F(z)}{1-i}$$

(ii) If $u - v$ is given, then let $F(z) = U + iV$ is a new analytic function whose real part $U = u - v$, then the required analytic function is

$$f(z) = \frac{F(z)}{1+i}$$

Example 4.41

If $f(z) = u + iv$ and $u - v = e^x(\cos y - \sin y)$, find the corresponding analytic function.

Solution:

Let $F(z) = U + iV$ be an analytic function such that

$$U = u - v = e^x(\cos y - \sin y) \tag{1}$$

$$\frac{\partial U}{\partial x}(x,y) = e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial x}(z,0) = e^z \tag{2}$$

$$\frac{\partial U}{\partial y}(x,y) = e^x(-\sin y - \cos y)$$

$$\frac{\partial U}{\partial x}(z,0) = e^z(-1) \tag{3}$$

$$\text{Now} \quad F'(z) = \frac{\partial U}{\partial x}(x,y) - i\frac{\partial U}{\partial x}(x,y)$$

$$= \frac{\partial U}{\partial x}(z,0) - i\frac{\partial U}{\partial y}(z,0) \qquad \text{(by M-Thomson method)}$$

$$\therefore \quad F(z) = \int \left[\frac{\partial U}{\partial x}(z,0) - i\frac{\partial U}{\partial y}(z,0)\right] dz + c$$

$$= \int [e^z + ie^z]dz + c \qquad \text{(from (1) and (2))}$$

$$= (1+i)e^z + c$$

Hence required analytic function is $f(z) = \dfrac{F(z)}{1+i} = e^z + A.$

Example 4.42

If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z) = u + iv$ is an analytic function, find $f(z)$. **(AU 2007)**

Solution:

$$\text{Given} \quad u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x}$$

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

We construct a new analytic function $F(z) = U + iV$.

$$\text{Such that} \quad U = u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\therefore \quad \frac{\partial U(x,y)}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial U(z,0)}{\partial x} = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = -\text{cosec}^2 z \qquad (1)$$

$$\frac{\partial U(x,y)}{\partial y} = \frac{2\sin 2x \sinh 2y}{(\cos 2y - \cos 2x)^2}$$

$$\frac{\partial U(z,0)}{\partial y} = 0 \qquad (2)$$

$$\text{Now} \quad F'(z) = \frac{\partial U(x,y)}{\partial x} - i\frac{\partial U(x,y)}{\partial y}$$

$$= \frac{\partial U(z,0)}{\partial x} - i\frac{\partial U(z,0)}{\partial y} \quad \text{(by Milne-Thomson method)}$$

$$\therefore \quad F'(z) = \int \left[\frac{\partial U}{\partial x}(z,0) - i\frac{\partial U}{\partial y}(z,0)\right] dz + c$$

$$= -\int \cos ec^2 z dz + c \quad \text{(from (1) and (2))}$$

$$= \cot z + c$$

Hence the required analytic function is

$$f(z) = \frac{F(z)}{1-i} = \frac{\cot z + c}{1-i} = \left(\frac{1+i}{2}\right)\cot z + A$$

Example 4.43

If $f(z) = u + iv$ is an analytic function such that $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ and $f(\pi/2) = 0$. Find $f(z)$.

Solution:

Let $F(z) = U + iV$ be a new analytic function such that

$$U = u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$$

$$\therefore \quad \frac{\partial U(x,y)}{\partial x} = \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) + (\cos x + \sin x - e^{-y})(2\sin x)}{(2\cos x - e^{y} - e^{-y})^2}$$

$$\frac{\partial U(z,0)}{\partial x} = \frac{1}{2}\,\text{cosec}^2\frac{z}{2} \qquad (1)$$

$$\frac{\partial U(x,y)}{\partial x} = \frac{(2\cos x - e^{y} - e^{-y})e^{-y} - (\cos x + \sin x - e^{-y})(-e^{y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2}$$

$$\frac{\partial U}{\partial y}(z,0) = -\frac{1}{2}\,\text{cosec}^2\frac{z}{2} \qquad (2)$$

$$
\begin{aligned}
F'(z) &= \frac{\partial U(x,y)}{\partial x} - i\frac{\partial U(x,y)}{\partial y} \\
&= \frac{\partial U(z,0)}{\partial x} - i\frac{\partial U(z-0)}{\partial y} \qquad \text{(by Milne-Thomson method)} \\
\therefore \quad F(z) &= \int \left[\frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)\right] dz + c \\
&= \int \frac{(1+i)}{2} \operatorname{cosec}^2 \frac{z}{2} dz + c \qquad \text{(from (1) and (2))} \\
&= \frac{(1+i)}{2} 2\cot\frac{z}{2} + c
\end{aligned}
$$

Hence the required analytic function is

$$f(z) = \frac{F(z)}{1+i} = -\cot\frac{z}{2} + A$$

Since $f(\pi/2) = 0 \quad \therefore \quad \cot(\pi/4) = c \quad \therefore \quad c = 1$

$$\therefore \quad f(z) = 1 - \cot(z/2)$$

Example 4.44

Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally when $w = \sin z$.

Solution:

$$
\begin{aligned}
\text{Given} \quad w = u + iv &= \sin z \\
&= \sin(x+iy) \\
&= -\sin x \cos iy + \cos x \sin iy \\
&= \sin x \cosh y + i \cos x \sinh y
\end{aligned}
$$

$$u = \sin x \cosh y = c_1 \tag{1}$$

$$\text{and} \quad v = \cos x \sinh y = c_2 \tag{2}$$

Differentiating (1) and (2) with respect to x, we get

$$\cos x \cosh y + \sin x \sinh y \left(\frac{dy}{dx}\right) = 0$$

$$\text{or} \quad \left(\frac{dy}{dx}\right) = m_1 = \frac{-\cos x \cosh y}{\sin x \sinh y}$$

$$\text{and} \quad -\sin x \sinh y + \cos x \cosh y \left(\frac{dy}{dx}\right) = 0$$

or $\dfrac{dy}{dx} = m_2 = \dfrac{\sin x \sinh y}{\cos x \cosh y}$

$$\therefore \quad m_1 m_2 = \left(\frac{-\cos x \cosh y}{\sin x \sinh y}\right)\left(\frac{\sin x \sinh y}{\cos x \cosh y}\right) = -1$$

Hence the two family of current $u = c_1$ and $v = c_2$ form an orthogonal system.

Example 4.45

If $f(z) = z^3$, show that u and v satisfy the C-R equations. Also prove that the families of curves $u = c_1$ and $v = c_2$ are orthogonal to each other. **(AU 2004)**

Solution:

$$\text{Given} \quad f(z) = z^3 = (x+iy)^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$\therefore \quad u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3$$

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad ; \quad \frac{\partial v}{\partial x} = -6xy$$

$$\frac{\partial u}{\partial y} = -6xy \quad ; \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\text{Clearly} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence the C-R equations are satisfied. Differentiating $u = c_1$ with respect to x, we get

$$3x^2 - 3\left(2xy\frac{dy}{dx} + y^2\right) = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1 \quad \text{(say)}$$

Now, differentiating $v = c_2$ with respect to x, we get,

$$3\left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = m_2 \quad \text{(say)}$$

$$\therefore \quad m_1 \times m_2 = \left(\frac{x^2 - y^2}{2xy}\right)\left(\frac{-2xy}{x^2 - y^2}\right) = -1$$

$\therefore$ The two families of curves are orthogonal.

Example 4.46

If $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2+y^2}$, prove that both u and v satisfy Laplace equation, but that $f(z) = u + iv$ is not a analytic function of z.

Solution:

Given $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2+y^2}$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y, \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$, i.e. u satisfies the Laplace equation

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial^2 v}{\partial y^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$\therefore \quad \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ i.e. v satisfies the Laplace equation

$$\text{But} \quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

The C-R equations are not satisfied. Hence $f(z) = u + iv$ is not an analytic function of z.

Example 4.47

If u & v are functions of x and y satisfying Laplace's equation, show that $p + iq$ is analytic where

$$p = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad \text{and} \quad q = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Solution:

Given that both u & v are harmonic

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{1}$$

Hence $f(z) = p + iq$ will be analytic if p & q satisfy C-R equations, namely

$$\therefore \quad \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \quad \text{(from 1)}$$

$$\therefore \quad \frac{\partial p}{\partial x} = -\frac{\partial q}{\partial y}$$

$$\text{Similarly} \quad \frac{\partial p}{\partial y} = \frac{-\partial q}{\partial x}$$

Hence $p + iq$ satisfies the C-R equation and hence it is an analytic function.

Example 4.48

If $f(z)$ is a regular function of z prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left|f'(z)\right|^2$$

(AU 2007, 2008, 2009, 2010, 2011)

Solution:

We know that

$$\left.\begin{array}{c} u_x = v_y \text{ and } u_y = -v_x \\ u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0 \end{array}\right\} \tag{1}$$

Let $f(z) = u + iv$ and $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$|f(z)|^2 = u^2 + v^2 \text{ and } |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)$$

$$= \frac{\partial^2 (u^2 + v^2)}{\partial x^2} + \frac{\partial^2 (u^2 + v^2)}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right]$$

$$= 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + v\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x}\right]$$

$$+ 2\left[u\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial y^2}\right]$$

$$= \left[\begin{array}{c} u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 \\ + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \end{array}\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{-\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] \quad \text{(using (1))}$$

$$= 2\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 4\left|f'(z)\right|^2 \quad \text{(using (2))}$$

Example 4.49

If $f(z)$ is an analytic function of z_1 show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^3 = 6u\left|f'(z)\right|^2$$

Solution: **(AU 2006)**

Let $f(z) = u + iv$, then $\left|f'(z)\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$. Since $f(z)$ is analytic

$$\left.\begin{array}{ll} \text{(i)} & \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \\ \text{(ii)} & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \end{array}\right\} \quad (1)$$

Now

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^3 = \frac{\partial^2(u^3)}{\partial x^2} + \frac{\partial^2(u^3)}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\left(3u^2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(3u^2\frac{\partial u}{\partial y}\right)$$

$$= 3\left[u^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.2u\frac{\partial u}{\partial x}\right] + 3\left[u^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot 2u\frac{\partial u}{\partial y}\right]$$

$$= 3\left[u^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2u\left(\frac{\partial u}{\partial x}\right)^2 + 2u\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 6u\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2\right] \quad \text{(using (1))}$$

$$= 6u|f'(z)|^2$$

Example 4.50

If $f(z) = u + iv$ is analytic, show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$. **(AU 2012)**

Solution:

Let $f(z) = u + iv$ be analytic

$$\log f(z) = \log|f(z)| + i \text{ amp } f(z)$$

since $f(z) \neq 0, \log|f(z)|$ exists. Further since $f(z)$ is analytic $\log f(z)$ is also analytic.

$\therefore$ $\log|f(z)|$ and amp $f(z)$ are the real and imaginary parts of the analytic function $\log f(z)$.

Hence both $\log|f(z)|$ and amp $f(z)$ satisfy Laplace equation.

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$$

4.5 Applications

Harmonic functions play an important role in the study of two dimensional steady flow such as fluid flow, electric current flow and heat flow, the paths of fluid particles are called *stream lines* and their orthogonal trajectories are called *equipotential lines*. When the flow is irrotational it can be shown that the harmonic function $\phi(x, y)$ such that velocity $\bar{v} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j}$ of the fluid is in the x and y directions. This function $\phi(x, y)$ is called the *velocity potential* of the motion.

The function $\psi(x, y)$ such that $\phi(x, y) + i\psi(x, y)$ is analytic is called *stream function*.

The analytic function $f(z) = \phi + i\psi$ is called the *complex potential* of the flow.

In heat flow problems, the curve $\phi = c_1$ & $\psi = c_2$ are respectively called *isothermals* and *heat flow lines*.

Example 4.51

In a complex electric field, the potential is $w = \phi + i\psi, \psi = x^3 - 3xy^2$. Find ϕ.

Solution:

Given $\psi = x^3 - 3xy^2$, is the imaginary part of an analytic function.

$$\therefore \quad w = \phi(x,y) + i\psi(x,y)$$

$$\therefore \quad \frac{\partial\psi(x,y)}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial\psi(z,0)}{\partial x} = 3z^2 \tag{1}$$

$$\frac{\partial\psi(x,y)}{\partial y} = -6xy$$

$$\frac{\partial\psi(z,0)}{\partial y} = 0 \tag{2}$$

$$\text{Now} \quad \frac{dw}{dz} = \frac{\partial\psi(x,y)}{\partial y} + i\frac{\partial\psi(x,y)}{\partial x} \quad \text{(by C-R equations)}$$

$$= \frac{\partial\psi(z,0)}{\partial y} + i\frac{\partial\psi(z,0)}{\partial x} \quad \text{(by Miline-Thomson rule)}$$

$$\therefore \quad w = \int\left[\frac{\partial\psi(z,0)}{\partial y} + i\frac{\partial\psi(z,0)}{\partial x}\right] dz + c.$$

$$= \int \left[0 + i3z^2\right] dz + c = iz^3 + c$$

$$w = iz^3 + c.$$

$$\text{but} \quad w = i(x+iy)^3 + c$$

$$= i[x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3] + c.$$

$$= i[x^3 - 3xy^2] + [-3x^2y + y^3] + c.$$

$$\therefore \quad \phi = y^3 - 3x^2y + c.$$

Example 4.52

An incompressible fluid flowing over the xy-plane has the velocity potential $\phi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$. Examine if this is possible and find a stream function ψ.

Solution:

Given $\phi(x,y) = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

$$\frac{\partial \phi}{\partial x} = 2x + \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2 - \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} \quad ; \quad \frac{\partial \phi(z,0)}{\partial x} = 2z - \frac{1}{z^2} \qquad (1)$$

$$\frac{\partial \phi}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial \phi(z,0)}{\partial y} = 0$$

$$\frac{\partial^2 \phi}{\partial y^2} = -2 - \frac{2x(x^2-3y^2)}{(x^2+y^2)^3} \qquad (2)$$

from (1) and (2)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\therefore$ ϕ satisfies Laplaces equation.

Hence it can be a possible form of the velocity potential function.

To find the stream function φ

Let $f(z) = \phi(x,y) + i\psi(x,y)$

$$\therefore \quad f'(z) = \frac{\partial \phi(x,y)}{\partial x} - i\frac{\partial \phi(x,y)}{\partial y} \quad [\ \phi \text{ is the real part}]$$

$$= \frac{\partial \phi(z,0)}{\partial x} - i\frac{\partial \phi(z,0)}{\partial y} \quad \text{(by Milne - Thomson rule)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial \phi(z,0)}{\partial x} - i\frac{\partial \phi(z,0)}{\partial y}\right] dz + c = \int \left[2z - \frac{1}{z^2}\right] dz + c$$

$$f(z) = z^2 + \frac{1}{z} + c$$

$$\therefore \quad f(z) = (x+iy)^2 + \frac{1}{x+iy} + c$$

$$= x^2 - y^2 + 2ixy + \frac{x-iy}{x^2+y^2} + c$$

$$= \left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + i\left(2xy - \frac{y}{x^2+y^2}\right) + c$$

Hence, the stream function is $\psi(x,y) = 2xy - \dfrac{y}{x^2+y^2} + c$.

Example 4.53

In a two dimensional fluid flow, if $xy(x^2-y^2)$ can represent the stream function. If so, find the corresponding velocity function and also the complex potential.

Solution:

Given stream function $\psi(x,y) = xy(x^2-y^2)$, which should be the imaginary part of an analytic function & hence harmonic.

$$\therefore \quad \psi = x^3y - xy^3$$

$$\frac{\partial \psi}{\partial x} = 3x^2y - y^3; \qquad \frac{\partial^2 \psi}{\partial x^2} = 6xy$$

$$\frac{\partial \psi}{\partial y} = x^3 - 3xy^2; \qquad \frac{\partial^2 \psi}{\partial y^2} = -6xy$$

$$\text{Clearly} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$\therefore$ ψ is a harmonic function, therefore it represents the stream function. Let $\phi(x,y)$ be the corresponding velocity potential.

Then $f(z) = \phi + i\psi$ is analytic.

$$\text{Now} \quad f'(z) = \frac{\partial \psi(x,y)}{\partial x} + i\frac{\partial \psi(x,y)}{\partial x}$$

$$= \frac{\partial \psi(x,y)}{\partial y} + i\frac{\partial \psi(x,y)}{\partial x} \qquad \text{(by C-R equations)}$$

$$= \frac{\partial \psi(z,0)}{\partial y} + i\frac{\partial \psi(z,0)}{\partial x} \qquad \text{(by Milne - Thomson rule)}$$

$$\therefore \quad f(z) = \int \left[\frac{\partial \psi(z,0)}{\partial y} + i\frac{\partial \psi(z,0)}{\partial x}\right] dz + c$$

$$= \int \left[z^3 + i(0)\right] \; dz + c$$

$$= \frac{z^4}{4} + c$$

$$= \frac{(x+iy)^4}{4} + c$$

$$= \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + c + i(x^3y - xy^2)$$

$$\therefore \quad \phi(x,y) = \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + c$$

Example 4.54

Find the corresponding complex potential $w = \phi + i\psi$ if $\phi = (x-y)(x^2+4xy+y^2)$ represents the equipotential for an electric field. Hence find ψ.

Solution:

If ϕ represents the equipotential for an electric field, it should be the real part of analytic function and hence harmonic.

$$\text{Given} \quad \phi = (x-y)(x^2+4xy+y^2)$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = (x-y)(2x+4y) + (x^2+4xy+y^2)$$

$$\frac{\partial^2 \phi}{\partial x^2} = (x-y)(2) + (2x+4y) + (2x+4y)$$

$$= 2x - 2y + 2x + 4y + 2x + 4y$$

$$= 6x + 6y \qquad (1)$$

$$\frac{\partial \phi}{\partial y} = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$$

$$\frac{\partial^2 \phi}{\partial y^2} = -(6x+6y) \qquad (2)$$

from (1) and (2)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Hence ϕ is harmonic and therefore it can represent the equipotential of an electric field. The corresponding complex potential

$$w = \phi + i\psi$$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x}(x,y) - i\frac{\partial \phi}{\partial y}(x,y) \quad [\phi \text{ is the real part}]$$

$$= \frac{\partial \phi}{\partial x}(x,0) - i\frac{\partial \phi}{\partial y}(z,0) \quad \text{(by Milne-Thomson Method)}$$

$$\therefore \quad w = \int \left[\frac{\partial \phi}{\partial x}(z,0) - i\frac{\partial \phi}{\partial y}(z,0)\right] dz + c$$

$$= \int [3z^2 - i3z^2] dz + c$$

$$= 3\int (1-i)z^2 dz + c$$

$$= (1-i)z^3 + c$$

$$\text{but} \quad w = \phi(x,y) + i\psi(x,y)$$

$$= (1-i)z^3 + c$$

$$= (1-i)(x+iy)^3 + c$$

$$= (x^3 - 3xy^2 + 3x^2y - y^3) + i(3x^2y - y^3 - x^2 + 3xy^2) + c$$

$$\text{Hence} \quad \psi = 3(x^2y + xy^2) - (x^3 + y^3) + c.$$

Exercise 4(a)

☞ **Part - A**

1. Define the continuity of a function of a complex variable.
2. Briefly explain the concept of limit of a function of a complex variable.
3. Differentiate between the limit of a function of a complex variable and of a real variable. **(AU 2012)**
4. Prove that $f(z) = Re(z)$ is not differentiable at any point.
5. Prove that $f(z) = Im(z)$ is not differentiable at any point.
6. When is a function of a complex variable said to be differentiable at a point.
7. Define an analytic function. **(AU 2010)**
8. State the Cauchy-Riemann equations. **(AU 2011)**
9. State the necessary condition for a function for to be analytic.
10. State the sufficient condition for a function $f(z)$ to be analytic.
11. State the Cauchy-Riemann equations in polar coordinates.
12. Prove that $f(z) = \dfrac{z-1}{z+1}$ is differentiable at every point $z \neq -1$ and find $f'(z)$.
13. Show that xy^2 cannot be real part of an analytic function.
14. For what values of z do the function w defined by the following equations cease to be analytic.
    (a) $z = e^{-y}(\text{cosu} + \text{isinu}), w = u + iv$
    (b) $z = \text{sinu cosv} + \text{icoshu sinv}, w = u + iv$
    (c) $z = \text{sinu coshv} + \text{icosu sinhv}, w = u + iv$
15. Show that $f(z) = \bar{z} = x - iy$ is not analytic function of z.
16. Show that $f(z) = xy + iy$ is continuous everywhere but not analytic.
17. Test whether the following functions are analytic or not
    (a) $f(z) = e^x(\cos y - \text{isin } y)$
    (b) $f(z) = e^{-x}(\sin y + \text{icos } y)$

(c) $f(z) = e^x + \dfrac{(1+i)}{2}$

(d) $f(z) = \dfrac{1+iy}{x^2+y^2}$

(e) $f(z) = \dfrac{z}{x^2+y^2}$

(f) $f(z) = \dfrac{1}{z^2+4z+3}$

18. Define a harmonic function and give an example.

19. What do we mean by conjugate harmonic?

20. Verify whether the following functions are harmonic

(a) xy^2 (b) xy

(c) $e^x \sin y$ (d) $y + e^x \cos y$

(e) $e^x \cos y$ (f) $2x(1-y)$

21. Find the analytic function $f(z) = u + iv$, given that

(a) $u = x$ (b) $v = xy$

(c) $u = 2x(1-y)$ (d) $u = e^x \cos y$

(e) $v = -arg z$ (f) $u = \cos x \cosh y$

(g) $v = \sinh x \sin y$ (h) $u = \dfrac{x}{x^2+y^2}$

22. Find whether the Cauchy - Riemann equations are satisfied for the following functions.

(a) $w = z^2$ (b) $w = \cos(x - iy)$

(c) $w = \dfrac{z-1}{z+1}$ (d) $w = x^2 + iy^2$

(e) $w = \dfrac{x+iy}{x^2+y^2}$ (f) $w = xy(y + ix)$

(g) $w = e^x(\cos y - i \sin y)$ (h) $w = e^{-x}(\cos y - i \sin y)$

23. If $f(z)$ is analytic, show that f(z) is a constant if

(a) Real part of $f(z)$ is a constant

(b) Modulus $f(z)$ is a constant

(c) Conjugate of $f(z)$ is analytic

24. Find the constants a, b, c and d so that the following are differentiable at every point.

    (a) $f(z) = ax^2 - by^2 + icxy$

    (b) $f(z) = x + ay - i(bx + cy)$

    (c) $f(z) = (x^2 + axy + by^2) + ic(x^2 + dxy + y^2)$

☞ **Part - B**

25. Show that the function $f(z) = \dfrac{xy(x-y)}{x^2+y^2}$ is continuous at the origin given that $f(0) = 0$.

26. Show that $\lim\limits_{z\to 0}\left[\dfrac{xy^2}{x^2+y^2}\right]$ does not exist, even though the function approaches the same limit along every straight line through the origin.

27. Show that the following function are discontinuous at the origin given that

    (a) $f(z) = \dfrac{xy}{x^2+y^2}$ and $f(0) = 0$

    (b) $f(z) = \dfrac{xy(x-y)}{x^3+y^3}$ and $f(0) = 0$

    (c) $f(z) = \dfrac{x^2y(y-x)}{(x^3+y^3)(x+y)}$ and $f(0) = 0$

    (d) $f(z) = \dfrac{x^3-y^3}{x^3+y^3}$ and $f(0) = 0$

    (e) $f(z) = \dfrac{x^2y}{x^4+y^2}$ $z \neq 0$ and $f(0) = 0$

    (f) $f(z) = \dfrac{xy}{2x^2+y^2}$ and $f(0) = 0$

28. Show that $f(z) = \dfrac{xy(x-y)}{x^2+y^2}$ is continuous at the origin given that $f(0) = 0$.

29. Discuss the continuity at the origin of

    (a) $f(z) = \dfrac{2xy}{\sqrt{x^2+y^2}}$ for every x and y excepting $(0, 0)$ and $f(0) = 0$.

    (b) $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$ and $f(0) = 0$.

(c) $f(z) = \dfrac{2xy(x+y)}{x^2+y^2}$ when x and y are not zero simultaneously and $f(0) = 0$.

30. Prove the following function are analytic and also find their derivatives using definition.

(a)	z^3	(b)	e^{-t}
(c)	$\sin z$	(d)	$\cosh z$
(e)	$\log z$	(f)	$z + 1/z$
(g)	$\tan z$	(h)	$1/z$

31. Show that the following functions are harmonic, find the corresponding analytic function

(a) $v = e^{-x}(x\cos y + y\sin y)$

(b) $u = e^{2x}(y\cos\ 2y + x\sin\ 2y)$

(c) $u = e^{-x}[(x^2 - y^2)\cos y + 2xy\sin y]$

(d) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

(e) $u = -\sin x \sinh y$ **(AU 2010)**

32. If $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2+y^2}$, prove that both u and v satisfy Laplace's equation but $u + iv$ is not a regular function of z.

33. Show that the function $u(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic and determine its conjugate.

34. Prove that $u(x, y) = x(x^2 - 3y^2) + (x^2 - y^2) + 2xy$ is harmonic, find the conjugate harmonic function v and hence $f(z)$.

35. Verify that the family of curve $u =$ constant and $v =$ constant cut orthogonally when $w = z^4$.

36. Given the function $f(z) = z^2$ show that u and v satisfy the C-R equations and Laplace equations and that the families of curve $u = c_1$ and $v = c_2$ are orthogonal to each other.

37. Show that the curve $r^n = \alpha \sec n\theta$ and $r^n = \beta$ cosecn θ intersect orthogonally.

38. If $f(z) = u + iv$ is analytic and $v = \dfrac{2\sin x \sin y}{\cos 2x + \cosh 2y}$, find u.

39. If $P + iQ$ is an analytic function of z and if $P = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} + 2\cos 2x}$, determine Q.

40. Find the analytic function $f(z) = u + iv$, if $u + v = \dfrac{2x}{x^2 + y^2}$ and $f(1) = i$.

41. Find the analytic function $f(z) = u + iv$ if $u = e^{-2xy}\sin(x^2 - y^2)$ and hence find v.

42. Find the analytic function $f(z) = u + iv$ if

    (a) $u - v = \dfrac{x - y}{x^2 + 4xy + y^2}$

    (b) $u - v = \dfrac{e^y - \cos x - \sin x}{\cosh y - \cos 2x}$ and $f(\pi - z) = \dfrac{3 - i}{2}$

    (c) $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

    (d) $u - v = e^x(\cos y - \sin y)$

43. If $f(z) = u + iv$ is an analytic function, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f'(z)^2||f(z)|^{p-2}$.

44. If $f(z) = u + iv$ is an analytic function, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|u|^p = p(p-1)|u|^{p-2}|f(z)|^2$.

45. If $w = \phi + i\psi$ represents the complex potential for an electric field and $\phi = 3x^2y - y^3$, find ψ.

46. If in a two dimensional flow fluid, the velocity potential $\varphi = x^2 - y^2$. Find the stream function ψ.

47. In a two dimensional fluid flow, the velocity potential is given by $\phi = x^4 - 6x^2y^2 + y^4$. Find the stream function ψ.

48. Show that the equation $x^2y - xy^3 + xy + x + y = c$ can represent the path of electric current flow in an electric field. Also find the complex electric potential and the equation of the potential lines.

49. Show that $\psi = x^2 - y^2 - 3x - 2y - 2xy$ can represent the stream function of an incompressible fluid flow . Also find the corresponding velocity potential and complex potential.

28. Show that the circles $|z| \geq 1$ map onto confocal ellipses under the transformation $w = z + 1/z$.

29. Show that $w = e^z$ transforms the region between the real axis and a line parallel to the real axis at $z = \pi i$ into the upper half of the w-plane.

30. Find the image of the rectangular region $0 \leq y \leq 1, -\pi \leq x \leq \pi$ and its boundary under the transformation $w = \sin z$.

31. Show that under the transformation $w = \cos z$, straight lines parallel to x-axis are transformed into confocal ellipse in the w-plane.

4.7 The Bilinear Transformation or the Mobius Transformation

The transformation defined by

$$w = \frac{az+b}{cz+d} \tag{1}$$

where a, b, c, d are complex constants such that $ad - bc \neq 0$ is called a *bilinear or linear fractional transformation*.

The transformation can be rewritten as

$$w = \frac{a}{c}\left(\frac{z+b/a}{z+d/c}\right)$$

Then if $\frac{b}{a} = \frac{d}{c}$ or if $ad - bc = 0$ then for every value z, we have same value of w and we say that w is a constant. The expression ad-bc is called determinant of the bilinear transformation.

Critical points

If $w = \frac{az+b}{cz+d}$ then $\frac{dw}{dz} = \frac{ad-bc}{(c+d)^2}$

Now if $z = -d/c$ then $\frac{dw}{dz} = \infty$ and if $z = \infty$ then $\frac{dw}{dz} = 0$. These points $z = -d/c$ and $z = \infty$ are the critical points where the conformal property does not hold good.

The *inverse* of the transformation $= \frac{az+b}{ct+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation.

Hence it is clear that for each $z \neq -d/c$ we have a value of w and for each $w \neq a/c$ there corresponds a value of z and the correspondence between w and z is one -one. The exceptional points $z = -d/c$ and $w = a/c$ are mapped into the points $w = \infty$ and $z = \infty$ respectively.

These points will not remain exceptions if we adjoin a new point called point at infinity denoted by ∞ to the complex plane and the complex plane in this case is called extended complex plane. Thus the critical point $z = \infty$ corresponds to the point $w = a/c$.

To discuss the transformation (1) in detail, we can express (1) as the combination of simple transformation discussed in the previous section.

When $c \neq 0$, (1) can be rewritten as

$$w = \frac{a}{c} + \left(\frac{bc - ad}{c}\right)\frac{1}{cz + d}$$

Make the substitution

$$w_1 = cz + d \tag{2}$$

$$w_2 = \frac{1}{w} \tag{3}$$

$$\text{Then} \qquad w = A + Bw_2 \tag{4}$$

$$\text{where} \qquad A = \frac{a}{c} \quad \text{and} \quad B = \frac{bc - ad}{c}$$

(2), (3), (4) represent three successive transformations that map z into w. Each of the transformations maps circles and straight lines into circles and straight lines.

Hence the bilinear transformation (1) maps circles and straight line onto circles and straight line in general

$$\text{if } c = 0 \ \ (1) \text{ becomes } w = \frac{a}{d}z + \frac{b}{d}; \ \ d \neq 0$$

is also of the type $w = Cz + D$; this transforms circles into circles.

Hence the bilinear transformation always maps circles into circles with lines as limiting cases.

■ **Note:** The transformation $w = \dfrac{az + b}{cz + d}$, has four arbitrary constants a, b, c, d which can be re-written as

$$w = \frac{a/dz + b/d}{c/dz + 1} = \frac{Az + B}{Cz + 1}$$

which has three effective arbitrary constants and hence there conditions are necessary to determine bilinear transformation.

In particular, three distinct points z_1, z_2, z_3 can be mapped into any three specified distinct points w_1, w_2, w_3 i.e.

$$w_1 = \frac{Az_1 + B}{Cz_1 + 1}, \qquad w_2 = \frac{Az_2 + B}{Cz_2 + 1} \quad \text{and} \quad w_3 = \frac{Az_3 + B}{Cz_3 + 1}$$

we get three equations in A, B, C, solving then at the values of A, B, C uniquely and hence the bilinear transformation is unique.

Fixed points

The points which coincide with their transformations are called *fixed or invariant points* of the transformation.

In other words the fixed points of the transformation $w = f(z)$ are obtained from the equation $z = f(z)$.

For example if $w = f(z) = z^2$.

Then the invariant points are given by, $z = z^2$ or $z^2 - z = 0$

$$\begin{aligned} \text{or} \quad z^2 - z &= 0 \\ \text{or} \quad z(z-1) &= 0 \\ \text{or} \quad z &= 0 \quad \text{and} \quad z = 1 \end{aligned}$$

$\therefore$ $z = 0$ and $z = 1$ are the invariant points.

Cross ratio

If z_1, z_2, z_3, z_4 are distinct points taken in the order, then $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ is called the *cross ratio* of these points of the z-plane. **(AU 2009)**

Theorem: 1

To prove that cross ratio remains invariant under a bilinear transformation.

Proof:

$$\text{Let} \quad w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \tag{1}$$

and let w_1, w_2, w_3, w_4 be the images of the four points z_1, z_2, z_3, z_4 under the bilinear transformation (1)

$$\begin{aligned} \text{Then} \quad w_1 - w_2 &= \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d} \\ &= \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \\ \text{and} \quad w_3 - w_4 &= \frac{(ad - bc)(z_3 - z_4)}{((cz_3 + d)(cz_4 + d)} \quad \text{etc.} \end{aligned}$$

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(ad - bc)^2(z_1 - z_2)(z_3 - z_4)}{(ad - bc)^2(z_1 - z_4)(z_3 - z_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$\therefore$ the cross ratios are preserved under a bilinear transformation.

$$\therefore \quad (w_1, w_2, w_3, w_4) = (z_1\ z_2\ z_3\ z_4)$$

Theorem: 2

The transformation $w = f(z)$ determined by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{1}$$

is a bilinear transformation which maps z_1, z_2, z_3 on w_1, w_2, w_3.

Proof:

(1) Can be written as

$$\begin{aligned}&(w-w_1)(w_2-w_3)(z_1-z_2)(z_3-z)\\ &= (z-z_1)(z_2-z_3)(w_1-w_2)(w_3-w)\end{aligned} \tag{2}$$

Now, if $z = z_1$ in (2), we get $w - w_1$, which implies that w_1 is the image of z_1. Similarly if we put $z = z_3$ in (2) we get $w = w_3$, showing that w_3 in the image of z_3.

If $z = z_2$, then from (2) we have

$$(w-w_1)(w_2-w_3)(z_1-z_2)(z_3-z_2) = (z_2-z_1)(z_2-z_3)(w_1-w_2)(w_3-w)$$

$$\text{or} \quad (w-w_1)(w_2-w_3) = (w_1-w_2)(w_3-w)$$

$$\text{or} \quad w(w_2-w_2+w_1-w_2) = w_3(w_1-w_2) + w_1(w_2-w_3)$$

$$\text{or} \quad w(w_1-w_3) = w_2(w_1-w_3)$$

$$\therefore \quad w = w_2$$

Hence the three points $z_1\ z_2\ z_3$ are mapped onto $w_1\ w_2\ w_3$ respectively which therefore is the required bilinear transformation.

Example 4.81

Find the bilinear transformation which maps the points $z_1 = i, z_2 = 0$ and $z_3 = -1$ into $w_1 = 1, w_2 = i$ and $w_3 = 0$ respectively. **(AU 2007)**

Solution:

We know that the required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{Hence} \quad \frac{(w-1)\ (i-0)}{(w-0)(i-1)} = \frac{(z+i)(0+1)}{(z+1)(0+i)}$$

$$\frac{w-1}{w} = \frac{(z+i)(i-1)}{(z+1)(-1)}$$

$$(z+i)(-1)w = -(w-1)(z+1)$$

$$w(iz-i) = z+1$$

$w = -i\left(\dfrac{z+1}{z-1}\right)$ is the required bilinear transformation.

Example 4.82

Find the bilinear transformation mapping the points $z_1 = 1, z_2 = i, z_3 = i$, $z_3 = -1$ into $w_1 = 2, w_2 = i, w_3 = -2$ respectively. **(AU 2009)**

Solution:

The bilinear transformation that maps $z_1\ z_2\ z_3$ into $w_1\ w_2\ w_3$ is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)\,(\mathrm{i}+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(z-1)(i+1)(i-2)}{(z+1)(i-1)(i+2)}$$

$$= \frac{(z-1)(-3-i)}{(z+1)(-3+i)}$$

$$= \frac{-3z+3-iz+i}{-3z-3+iz+i}$$

$$\frac{(w-2)+(w+2)}{(w-2)-(w+2)} = \frac{(-3z+3-iz+i)+(-3z-3+iz+i)}{(-3z+3-iz+i)-(-3z-3+iz+i)}$$

$$\text{or}\quad \frac{2w}{-4} = \frac{-6z+2i}{-2iz+6} \qquad \left[\because\ \frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a+b}{a-b}=\frac{c+d}{c-d}\right]$$

$$\text{or}\quad \frac{w}{-2} = \frac{3z-i}{iz-3}$$

$\therefore\ w = \dfrac{-6z+2i}{iz-3}$ is the required transformation.

Example 4.83

Find the Bilinear transformation which maps $z = 0, z = 1, z = \infty$ into the points $w = i, w = 1, w = -i$. **(AU 2013)**

Solution:

$$(w,\ w_1,\ w_2,\ w_3) = (z,\ z_1,\ z_2,\ z_3)$$

$$\text{(i.e.,)}\qquad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad (1)$$

Take $z_1 = 0;\ z_2 = 1;\ z_3 = \infty;\ w_1 = i;\ w_2 = 1;\ w_3 = -i;$

To avoid the substitution of $z_3 = \infty$;.

Put $z_3 = \dfrac{1}{z_3'}$, and then Put $z_3' = 0$

$$(1) \qquad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(z_2 - \dfrac{1}{z_3'}\right)}{\left(z - \dfrac{1}{z_3'}\right)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2 z_3' - 1)}{(z_2 z_3' - 1)(z_2-z_1)}$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{z(-1)}{(-1)\cdot 1}$$

$$\frac{(w-i)}{(w+i)} = \frac{z(1-i)}{(1+i)}$$

$$\frac{2w}{2i} = \frac{z(1-i)+(1+i)}{(1+i)-(1-i)z}$$

$$\boxed{w = \frac{z+i}{iz+1}}$$

Example 4.84

What is the bilinear transformation which maps the points $z_1 = 1,\ z_2 = 0,\ z_3 = 1$ into the points $w_1 = 0$ $w_2 = i$ and $w_3 = 3i$? **(AU 2008)**

Solution:

The bilinear transformation that maps z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)\,(i-3i)}{(w-3i)(i-0)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}$$

$$\frac{-2iw}{(w-3i)i} = \frac{-(z+1)}{z-1}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

Therefore $\dfrac{(2w)+(w-3i)}{(2w)-(w-3i)} = \dfrac{(z+1)+(z-1)}{(z+1)-(z-1)}$ $\qquad \left[\because \ \dfrac{a}{b} = \dfrac{c}{d} \Rightarrow \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}\right]$

$$\frac{3w-3i}{w+3i} = \frac{2z}{2}$$

or $(3w-3i) = z(w+3i)$

$$3w - zw = 3zi + 3i$$

$$w(3-z) = 3i(z+1)$$

$$\therefore\; w = \frac{3i(z+1)}{3-z}$$ is the required bilinear transformation.

Example 4.85

Find the linear fractional transformation that maps the points $z = -i, 0, i$ in to the points $w = -1, i, 1$ respectively. **(AU 2010)**

Solution:

The bilinear transformation or linear fractional transformation that maps the point z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Let $z_1 = i, z_2 = 0, z_3 = i$ and $w_1 = -1, w_2 = i, w_3 = 1$

$$\therefore\quad \frac{(w+1)(i-1)}{(w-1)(i+1)} = \frac{(z+i)(0-i)}{(z-i)(0+i)}$$

$$\frac{w+1}{w-1} = \frac{-(-1+i)+(i+z)}{(iz-z+1+i)}$$

Therefore $$\frac{(1+w)+(1-w)}{(1+w)+(1-w)} = \frac{(-1+z+i+iz)+(1-z+i+iz)}{(-1+z+i+iz)-(1-z+i+iz)}$$

$$\frac{2w}{2} = \frac{-2+2z}{2i+2iz} \qquad \left[\because\; \tfrac{a}{b} = \tfrac{c}{d} \Rightarrow \tfrac{a+b}{a-b} = \tfrac{c+d}{c-d}\right]$$

$$\therefore\; w = \frac{i(1-z)}{1+z}$$ is the required transformation.

Example 4.86

Find the bilinear transformation which maps the points $z = 0, \; -i, \; -1$ into $w = i, \; 1, \; 0$ respectively. **(AU 2009)**

Solution:

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \quad \frac{(w-i)(1)}{(i-1)(-w)} = \frac{z(-i+1)}{i(-1-z)}$$

$$\begin{aligned}
\Rightarrow \quad \frac{w-i}{w(-i+1)} &= \frac{z(-i+1)}{-i(1+z)} \\
(w-i)(-i)(1+z) &= zw(-i+1)^2 \\
(-iw-1)(1+z) &= aw(1-1-2i) \\
-iw-1-izw-z &= -2izw \\
-1-z &= -izw+iw \\
-1-z &= iw(1-z)
\end{aligned}$$

$$\therefore \qquad w = \frac{-1-z}{i(1-z)+(1+i)} = -i\left(\frac{z+1}{z-1}\right)$$

Example 4.87

Find the Mobius transformation that maps the points $2, i, -2$ of the z-plane into the points $1, i, -1$ of the w-plane. **(AU 2011)**

Solution:

Let $z_1 = 2, z_2 = i, z_3 = -2$ and $w_1 = 1, w_2 = i, w_3 = -1$.

The Mobius transformation that maps the points z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\begin{aligned}
\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} &= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \\
\frac{(w-1)\,(i+1)}{(w+1)(i-1)} &= \frac{(z-2)(i+2)}{(z+2)(i-2)} \\
\frac{(w-1)}{(w+1)} &= \frac{3z-6+2i-zi}{3z+6+2i+iz} \qquad \left[\because \quad \tfrac{a}{b} = \tfrac{c}{d} \Rightarrow \tfrac{a+b}{a-b} = \tfrac{c+d}{c-d}\ \right]
\end{aligned}$$

Therefore
$$\begin{aligned}
\frac{(w-1)+(w+1)}{(w+1)-(w+1)} &= \frac{(3z-6+2i-zi)+(3z+6+2i+i)}{(3z-6+2i-zi)-(3z+6+2i+zi)} \\
\frac{2w}{-2} &= \frac{6z+4i}{-12-2zi}
\end{aligned}$$

$\therefore\ w = \dfrac{3z+2i}{zi+6}$ is the required Mobius transformation.

Example 4.88

Find the bilinear transformation that maps the points $z_1 = 1+i, z_2 = -i$, $z_3 = 2-i$ into $w_1 = 0, w_2 = 1, w_3 = i$ respectively. **(AU 2010, 2011)**

Solution:

The bilinear transformation that maps the points z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)\,(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)\,(1-i)}{(w-i)(1-0)} = \frac{(z-1-i)(-i-(2-i))}{(z-2+i)(-1-2i)}$$

$$\frac{w}{w-i} = \frac{2(z-1-i)}{3z+iz-7+i} = \frac{2z-2-2i}{3z+iz-7+i} \qquad \left[\because \tfrac{a}{b}=\tfrac{c}{d} \Rightarrow \tfrac{a+b}{a-b}=\tfrac{c+d}{c-d}\right]$$

$$\therefore \quad \frac{(w)+(w-i)}{(w)-(w-i)} = \frac{(2z-2-2i)+(3z+iz-7+i)}{(2z-2-2i)-(3z+iz-7+i)}$$

$$\frac{2w-i}{+i} = \frac{5z+iz-9-i}{-z-iz+5-3i}$$

$$\text{or} \qquad 2w = \frac{5zi-z-9i+1}{-z-iz+5-3i}+i$$

$$= \frac{4zi-4i+4}{-(1+i)z+5-3i}$$

$$\therefore \quad w = \frac{2zi-2i+2}{-(1+i)z+5-3i}$$

is the required bilinear transformation.

Example 4.89

Find the bilinear transformation which maps $-1, 0, 1$ of the z-plane into $-1, -i, 1$ of the w-plane. Show that under this transformation the upper half of the z-plane maps onto the interior of the unit circle $|w| = 1$. **(AU 2009, 2011)**

Solution:

Let $z_1 = -1 \quad z_2 = 0 \quad z_3 = 1$

and $w_1 = -1 \quad w_2 = -i \quad w_3 = 1$

The bilinear transformation that maps the points z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}$$

$$\frac{w+1}{w-1} = \frac{(z+1)(-1)(-i+1)}{(z-1)(-1)(i+1)}$$

$$= \frac{(z+1)(-i+1)}{(z-1)(i+1)} = \frac{-iz-i+z+1}{iz-i+z-1} \qquad \left[\because \tfrac{a}{b}=\tfrac{c}{d} \Rightarrow \tfrac{a+b}{a-b}=\tfrac{c+d}{c-d}\right]$$

Therefore

$$\frac{(w+1)+(w-1)}{(w+1)-(w-1)} = \frac{(-iz-i+z+1)+(iz-i+z-1)}{(-iz-i+z+1)-(iz-i+z-1)}$$

$$\frac{2w}{2} = \frac{-2i+2z}{-2iz+2}$$

$$\text{or} \qquad w = \frac{z-i}{-iz+1} = i\left(\frac{z-i}{z+i}\right)$$

is the required bilinear transformation.

The equation of the upper half of the w-plane is $Im(w) > 0$.

$$\therefore \quad Im(w) > 0 \Rightarrow Im\left[i\left(\frac{z-i}{z+i}\right)\right] > 0$$

$$\Rightarrow \text{Re}\left[\frac{z-i}{z+i}\right] < 0$$

$$\Rightarrow \text{Re}\left[\frac{(z-i)(\bar{z}+\bar{i})}{|z+i|^2}\right] < 0$$

$$\Rightarrow \text{Re}(z-i) < 0$$

$$\Rightarrow \text{Re}(z-i(z+)-1) < 0$$

$$\Rightarrow \text{Re}(z) - 1 < 0$$

$$\Rightarrow |z|^2 < 1$$

$$\Rightarrow |z| < 1$$

which represents the interior of the unit circle in the z-plane.

Hence under the transformation the upper half plane is mapped into the interior of the unit circle.

Example 4.90

Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Also find the images of

(i) $|z| < 1$

(ii) concentric circles $|z| = r, r > 1$. **(AU 2010)**

Solution:

Let $z_1 = 1 \quad z_2 = i \quad z_3 = -1$

and $w_1 = i \quad w_2 = 0 \quad w_3 = -i$

The bilinear transformation that maps the points z_1, z_2, z_3 onto w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w-i}{w+i} = \frac{-(z-1)(i+1)}{(z+1)(i-1)} \qquad \left[\because \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}\right]$$

Therefore

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = -\left[\frac{(z-1)(i+1)+(z+1)(i-1)}{(z-1)(i+1)-(z+1)(i-1)}\right]$$

$$\frac{2w}{-2i} = \frac{-(zi+1)}{i+z}$$

$$\text{or} \qquad w = \frac{i-z}{i+z}$$

is the required bilinear transformation.

(i) Consider $|z| \leq 1$

From the above transformation we get

$$z = i\left(\frac{1-w}{1+w}\right)$$

$$\therefore \quad |z| \leq 1 \Rightarrow \left|i\frac{1-w}{1+w}\right| \leq 1$$

$$\Rightarrow |1-w| \leq |1+w|$$

$$\Rightarrow (1-w)(1-\bar{w}) \leq (1+w)(1+\bar{w})$$

$$\Rightarrow 2(w+\bar{w}) \geq 0$$

$$\Rightarrow 4u \geq 0$$

$$\Rightarrow u \geq 0$$

$$\Rightarrow \text{Re}(w) \geq 0$$

which represents the real part of the w-plane.

Hence under the transformation, the interior of the circle.

$|z| = 1$ is mapped onto right half of the w-plane.

(ii) Now consider $|z| = r$ or $\left|i\frac{1-w}{1+w}\right| = r$

$$\text{or} \qquad |1-w|^2 = r^2|1+w|^2$$

$$\text{or} \qquad (1-w)(1-\bar{w}) = r^2(1+w)(1+\bar{w})$$

$$\text{or} \qquad w\bar{w} - \left(\frac{1+r^2}{1-r^2}\right)(w+\bar{w}) + 1 = 0$$

$$\text{or} \qquad (u^2+v^2) - \left(\frac{1+r^2}{1-r^2}\right)2u + 1 = 0$$

which represents a circle in the w-plane with centre at $\left(\dfrac{1+r^2}{1-r^2}, 0\right)$ and radius $\dfrac{2r}{1-r^2}$.

Hence under the transformation, $|z| = r$ is transformed to a circle with centre at $\left(\dfrac{1+r^2}{1-r^2}, 0\right)$ and radius $\dfrac{2r}{1-r^2}$ in the w-plane.

Example 4.91

Find the bilinear transformation that maps the points $z_1 = 0, z_2 = 1, z_3 = \infty$, into the points $w_1 = -1, w_2 = -2, w_3 = -i$ respectively. **(AU 2007)**

Solution:

The bilinear transformation that maps the points z_1, z_2, z_3 onto w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

since $z_3 = \infty$, we avoid this substitution and put $z_3 = \dfrac{1}{z_3}$

$\therefore$ when $z_3 = \infty \quad z_3' = 0.$

$$\therefore \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2z_3'-1)}{(zz_3'-1)(z_2-z_1)}$$

$$\frac{(w+1)(-2+i)}{(w+i)(-2+1)} = \frac{(z-0)(0-1)}{(0-1)(-1-0)}$$

$$\frac{w+i}{w-i} = \frac{(-z)(-1)}{1(-2+i)} = \frac{z}{-2+i}$$

$$\text{or} \quad (w+1)(-2+i) = z(w-i)$$

$$\text{or} \quad -2w-2+wi+i = zw-zi$$

$$\text{or} \quad (-2+i-z)w = -zi+2-i$$

$$\text{or} \quad w = \frac{-iz+2-i}{-z-2+i}$$

is the required bilinear transformation.

Example 4.92

Find the bilinear transformation that maps the points $z = 0, 1, \infty$ into $w = i, 1, -i$ respectively. **(AU 2012)**

Solution:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$z_1 = 0,\ z_2 = 1,\ \ z_3 = \frac{1}{z_3'} \text{ where } z_3' = 0$$

$$w_1 = i,\ w_2 = 1,\ \ w_3 = -i$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{(z-0)(1-1/z_x')}{\left(z-\frac{1}{z_3'}(1-0)\right)} = \frac{z(z_3'-1)}{z_3'} \times \frac{z_3'}{(zz_3'-1)(1)}$$

$$= \frac{zz_3'-z}{zz_3'-1} = \frac{-z}{-1} = \frac{z}{1}$$

$$\frac{w-i}{w+i} = \frac{(1-i)z}{(1+i)1} = \frac{z-zi}{1+i}$$

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{(z-zi)+(1+i)}{(z-zi)-(1+i)} \quad \left[\frac{a}{b} = \frac{c}{d} = \frac{a+b}{a-b} = \frac{c+d}{c-d}\right]$$

$$\frac{2w}{-2i} = \frac{z(1-i)+(1+i)}{z(1-i)-(1+i)}$$

$$\frac{-w}{i} = \frac{z+\frac{1+i}{1-i}}{z-\frac{1+i}{1-i}} = \frac{z+\frac{(1+i)^2}{i^2-i^2}}{z-\frac{(1+i)^2}{1^2-i^2}} = \frac{z+\frac{1+2i-1}{1+1}}{z-\frac{1+2i-1}{2}} = \frac{z+i}{z-i}$$

$$w = \frac{-i(z+i)}{z-i} = \frac{-iz-i^2}{z-i} = \frac{-iz+1}{z-i} = \frac{-i^2z+i}{iz-i^2} = \frac{z+i}{iz+1}$$

Example 4.93

Find the bilinear transformation that maps the points $z = \infty, i, 0$ into the points $w = 0, i, \infty$. **(AU 2004)**

Solution:

Let $z_1 = \infty \quad z_2 = i \quad z_3 = 0$

and $w_1 = 0 \quad w_2 = i \quad w_3 = \infty$

The bilinear transformation that maps the points z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Since $z_1 = \infty$ put $z_1 = \frac{1}{z_1'}$ where $z_1' = 0$

and $w_3 = \infty$ put $w_3 = \frac{1}{w_3'}$ where $w_3' = 0$

$$\text{Here} \quad \frac{(w-w_1)(w_2w_3'-1)}{(ww_3'-1)(w_2-w_1)} = \frac{(zz_1'-1)(z_2-z_3)}{(z-z_3)(z_2z_1'-1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$\therefore \quad w = \dfrac{-1}{z}$ is the required bilinear transformation.

Example 4.94

Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $\omega = i, 0, \infty$. Also find the image of the unit circle $|z| = 1$. **(AU 2009)**

Solution:

Given $z_1 = 0 \quad z_2 = -1 \quad z_3 = i$

$w_1 = i \quad w_2 = 0 \quad w_3 = \infty$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \tag{1}$$

$$\frac{(w-w_1)w_3\left(\dfrac{w_2}{w_3}-1\right)}{(w_1-w_2)w_3\left(1-\dfrac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(-1)}{(i)(1)} = \frac{(z)(-1-i)}{i\cdot(i-z)}$$

$$\frac{(w-i)}{i} = \frac{z(i+1)}{i-z}$$

$$(w-i)(i-z) = z(-1+i)$$

$$wi - zw + 1 + zi - z(-1+i) = 0$$

$$z\left[-w+i-(-1+i)\right] = -wi-1$$

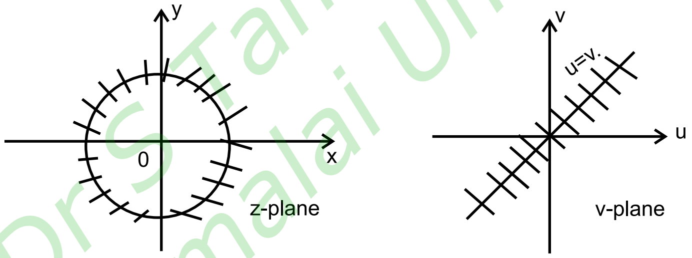
$$z(-w+i+1-i) = -wi-1$$

$$z = \frac{-wi-1}{-w+1} = \frac{iw+1}{w-1}$$

$$w = \frac{z+1}{z-i}$$

$$\text{Unit Circle } |z| = 1$$
$$\left|\frac{iw+1}{w-1}\right| = 1$$
$$|iw+1| = |w-1|$$
$$|i(u+iv)+1| = |u+iv-1|$$
$$|(1-v)+iu| = |(u-1)+iv|$$
$$u^2+(1-v)^2 = (u-1)^2+v^2$$
$$u^2+1+v^2-2v = u^2+1-2u+v^2$$
$$2u-2v = 0$$
$$u = v$$

The straight lines passes through origin in the w-plane as shown in the figure.



Example 4.95

Find the Mobius transformation which sends the points $z = 0, -i, 2i$ into the points $w = 5i, \infty, i/3$ respectively. **(AU 2007)**

Solution:

Let $z_1 = 0 \quad z_2 = -i \quad z_3 = 2i$

and $w_1 = 5i \quad w_2 = \infty \quad w_3 = i/3$

The bilinear transformation that maps the points z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Since $w_2 = \infty$ put $w_2 = \dfrac{1}{w_2}$ where $w_2' = 0$.

$$\therefore \quad \frac{(w-w_1)(1-w_2'w_3)}{(w-w_3)(1-w_2'w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-5i)(1-0)}{(w-i/3)(1-0)} = \frac{(z-0)(-i-2i)}{(z-2i)(-i-0)}$$

$$\frac{w-5i}{w-i/3} = \frac{-3zi}{-i(z-2i)} = \frac{3z}{z-2i}$$

Therefore
$$\frac{(w-5i)+(w-i/3)}{(w-5i)-(w-i/3)} = \frac{(3z)+(z+2i)}{(3z)-(z+2i)}$$

$$\frac{6w-16i}{-14i} = \frac{4z+2i}{2z-2i}$$

$$\frac{3w-8i}{-7i} = \frac{2z+i}{z-i}$$

Hence $w = \dfrac{-2z+5i}{-iz+1}$ is the required bilinear transformation.

Example 4.96

Find the bilinear transformation which maps the points $z = 0,\ 1\ ,\infty$ into $w = i,\ 1,\ -i$ respectively. **(AU 2010)**

Solution:

$$(w,\ w_1,\ w_2,\ w_3) = (z,\ z_1,\ z_2,\ z_3)$$

(i.e.,)
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad (1)$$

Take $z_1 = 0; z_2 = 1; z_3 = \infty; w_1 = i; w_2 = 1; w_3 = -i;$

To avoid the substitution of $z_3 = \infty$;.

Put $z_3 = \dfrac{1}{z_3'}$, and then Put $z_3' = 0$

(1)
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(z_2-\frac{1}{z_3'}\right)}{\left(z-\frac{1}{z_3'}\right)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2z_3'-1)}{(z_2z_3'-1)(z_2-z_1)}$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{z(-1)}{(-1)\cdot 1}$$

$$\frac{(w-i)}{(w+i)} = \frac{z(1-i)}{(1+i)}$$

$$\frac{2w}{2i} = \frac{z(1-i)+(1+i)}{(1+i)-(1-i)z}$$

$$\boxed{w = \frac{z+i}{iz+1}}$$

Example 4.97

Find the bilinear transformation that maps the point $0, -1, i$ of the z-plane into $i, 0, \infty$ of the w-plane. **(AU 2008)**

Solution:

Let $z_1 = 0 \quad z_2 = -1 \quad z_3 = i$
and $w_1 = i \quad w_2 = 0 \quad w_3 = \infty$

The bilinear transformation that maps the points z_1, z_2, z_3 into w_1, w_2, w_2 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Since $w_3 = \infty$ put $w_3 = \frac{1}{w_3'}$ where $w_3' = 0$

$$\therefore \quad \frac{(w-w_1)(w_2w_3'-1)}{(ww_3'-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0-1)}{(0-1)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\frac{(w-i)(-1)}{i} = \frac{-z(1+i)}{(z-i)(-1)}$$

$$w - i = \frac{-zi(1+i)}{z-i} = \frac{-zi+z}{z-i}$$

$$\therefore \quad w = \frac{-zi+z}{z-i} + i = -\frac{zi+z+zi+1}{z-i}$$

or $w = \dfrac{z+1}{z-i}$ is the required bilinear transformation.

Example 4.98

Find the bilinear transformation which maps $z = 1, 0, -1$ into $\omega = 0, -1, \infty$ respectively. What are the invariant points of the transformation? **(AU 2010)**

Solution:

The bilinear transformation that maps z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{Since } w_3 = \infty, w_3' = 0 \qquad \left(w_3 = \frac{1}{w^3}\right)$$

$$\therefore \quad \frac{(w-w_1)(w_2w_3'-1)}{(ww_3'-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(-1-0)} = \frac{(z-1)(0+1)}{(z+1)(0-1)}$$

$$\frac{-w}{1} = \frac{(z-1)}{-(z+1)}$$

$$w = \frac{z-1}{z+1}$$

is the required bilinear transformation.

Example 4.99

Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$ and find the fixed points of this transformation. **(AU 2009)**

Solution:

Given $z_1 = 1 \quad z_2 = i \quad z_3 = -1$

$w_1 = i \quad w_2 = 0 \quad w_3 = -i$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \quad \frac{[(w-i)(0-(-i))]}{[(w-(-i))(0-(i))]} = \frac{(z-1)(i-(-1))}{[(z-(-1))]\,(i-1)}$$

$$\Rightarrow \quad \frac{(w-i)(i)}{(w+i)(-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \quad -\frac{w-i}{w+i} = \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} \cdot \frac{i+1}{i+1}$$

$$\Rightarrow \quad -\frac{w-i}{w+i} = \frac{z-1}{z+1} \cdot \left(\frac{-1+i+i+1}{-2}\right)$$

$$\Rightarrow \quad -\frac{w-i}{w+i} = \frac{z-1}{z+1} \cdot \left(\frac{2i}{-2}\right)$$

$$\Rightarrow \quad \frac{w-i}{w+i} = i \cdot \frac{z-1}{z+1}$$

$$\Rightarrow \quad \frac{w-i}{w+i} = \frac{zi-i}{z+1}$$

$$
\begin{aligned}
wz + w - iz - i &= wzi + zi^2 - wi - i^2 \\
wz + w - iz - i &= wzi - z - wi + 1 \\
wz + w - iz - i - wzi + z + wi - 1 &= 0 \\
w[z + 1 - zi + i] &= i + iz + 1 - z \\
\therefore \qquad w &= \frac{(1 - z) + i(z + 1)}{z(1 - i) + (1 + i)}
\end{aligned}
$$

Example 4.100

Find the invariant points of the transformation

(i) $w = \dfrac{6z - 9}{z}$ (ii) $w = \dfrac{2z + 4i}{iz + 1}$

(iii) $w = \dfrac{z - 1}{z + 1}$ (iv) $w = \dfrac{(2 + i)z - 2}{z + i}$

Solution:

If z_1 is a fixed or invariant point of the transformation, then z_1 is transformed into

(i) Consider

$$
\begin{aligned}
w &= \frac{6z - 9}{z} \\
\therefore \quad z_1 &= \frac{6z_1 - 9}{z_1} \\
\text{or} \qquad z_1^2 - 6z_1 + 9 &= 0 \\
(z_1 - 3)^2 &= 0
\end{aligned}
$$

$\therefore$ $3, 3$ are the invariant points of the transformation.

(ii) Now

$$
\begin{aligned}
w &= -\frac{2z + 4i}{iz + 1} \\
z_1 &= -\frac{2z_1 + 4i}{iz_1 + 1} \\
\text{or} \qquad iz_1^2 + z_1 &= -2z_1 - 4i \\
z_1^2 - 3iz_1 + 4 &= 0 \\
(z_1 - 4i)(z + i) &= 0 \\
\therefore \qquad z_1 &= 4i, -i
\end{aligned}
$$

Hence $4i$ and $-i$ are the invariant points of the transformation.

(iii)
$$w = \frac{z-1}{z+1}$$
$$\therefore \quad z_1 = \frac{z_1 - 1}{z_1 + 1}$$
$$\text{or} \quad z_1(z_1+1) - (z_1 - 1) = 0$$
$$\text{or} \quad z_1^2 + 1 = 0$$
$$\text{or} \quad z_1 = \pm i$$

Hence $+i$ and $-i$ are the invariant points of the transformation.

(iv)
$$w = \frac{(2+i)z - 2}{z+i}$$
$$\therefore \quad z_1 = \frac{(2+i)z_1 - 2}{z_1 + i}$$
$$\text{or} \quad z_1(z_1+i) - (2+i)z_1 + 2 = 0$$
$$\text{or} \quad z_1^2 - 2z_1 + 2 = 0$$
$$\text{or} \quad z = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Hence $1+i$ and $1-i$ and the invariant points of the transformation.

Example 4.101

Find the invariant points of the transformation $w = \dfrac{2z+6}{z+7}$. **(AU 2009)**

Solution:

The invariant points of the transformation is given by
$$z = \frac{2z+6}{z+7}$$
$$z^2 + 7z - 2z - 6 = 0$$
$$z^2 + 5z - 6 = 0$$
$$z^2 - z + 6z - 6 = 0$$
$$z(z-1) + 6(z-1) = 0 \quad \Rightarrow \quad (z-1) = 0 \quad \text{(or)} \quad z+6 = 0$$
$$\Rightarrow \quad \boxed{z = 1 \quad \text{(or)} \quad z = -6}$$

Example 4.102

Prove that a linear transformation has almost two fixed points. **(AU 2012)**

Solution:

$$\text{Take } w = \frac{az+b}{cz+d}, \text{ a bilinear transformation}$$
$$\text{Put } z = \frac{az+b}{cz+d}, \text{ to get fixed points}$$
$$\text{i.e., } z(cz+d) = az+b, \text{ which is a quadratic in } z$$

$\therefore$ There will be almost two fixed points.

Example 4.103

Find the fixed points of mapping $w = \dfrac{6z-9}{z}$. **(AU 2013)**

Solution:

The find points are given by $z = \dfrac{6z-9}{z}$

$$\text{i.e., } z^2 = 6z - 9$$
$$z^2 - 6z + 98 = 0$$
$$(z-3)(z-3) = 0$$
$$\text{i.e., } z = 3, 3$$

Exercise 4(c)

☞ **Part - A**

1. Define a bilinear transformation. **(AU 2007)**
2. Define Mobius transformation. **(AU 2008)**
3. Define the cross ratio of four points in a complex plane. **(AU 2008)**
4. What are fixed points of a transformation?
5. What is Schwarz - Christoffel transformation? Also write the formula.
6. State the cross - ratio property of a bilinear transformation. **(AU 2008)**
7. Find the invariant points of the transformation $w = z^3$. **(AU 2010)**
8. What are invariant points of a transformation?
9. What does a general bilinear transformation transform a circle into.

10. What are the critical points of the bilinear transformation $w = \dfrac{az+b}{cz+d}$, if $ad - bc \neq 0$?

11. Define the determinant of a bilinear transformation.

12. Find the value of the determinant of the transformation $w = \dfrac{1+iz}{z+i}$.

13. Find the invariant points of the transformation

    (i) $w = \dfrac{3z-4}{z-1}$ (ii) $w = \dfrac{(2+i)z-2}{z-0}$ (iii) $w = \dfrac{z-3}{z+1}$

    (iv) $w = \dfrac{z-1-i}{z+2}$ (v) $w = \dfrac{5t-5i}{iz-1}$

14. In general, how many invariant points does a bilinear transformation have?

15. Find the condition for the fixed points of the transformation $w = \dfrac{az+b}{cz+d}$ to be equal.

16. Find all the bilinear transformation without fixed points in the finite plane.

17. Find all the bilinear transformations whose invariant points are ± 1.

18. Find all the bilinear transformations whose invariants points are $\pm i$.

19. Find any bilinear transformation having as the only invariant point.

☞ **Part - B**

21. Find out bilinear transformation which maps the points

    (i) $z_1 = 1, z_2 = i, z_3 = -1$ onto $w_1 = i, w_2 = 0, w_3 = -i$

    (ii) $z_1 = -i, z_2 = 0, z_3 = i$ onto $w_1 = -1, w_2 = i, w_3 = 1$

    (iii) $z = 1, -i, 0$ onto $w = 0, 2, -i$

22. Find the bilinear transformation that maps the points

    (i) $z_1 = \infty$ $z_2 = i, z_3 = 0$ onto $w_1 = 0, w_2 = i, w_3 = \infty$

    (ii) $z = -1, 1, \infty$ onto $w = -i - 1, i$

    (iii) $z = 1, -1, \infty$ onto $w = 1+i, 1-i, 1$

    (iv) $z = 1, i, -1$ onto $w = 0, 1, \infty$